Solutions to Quiz 8.

#1. \[ \lim_{x \to \infty} \frac{4 + 8\sqrt{x} + 3x}{4x + 8\sqrt{x} + 3} \]

Step #1.) Divide through by the highest power of \( x \) in the denominator.

\[ \lim_{x \to \infty} \frac{4\sqrt{x} + 8\frac{x}{\sqrt{x}} + 3x}{4x + 8\frac{x}{\sqrt{x}} + 3x} = \lim_{x \to \infty} \frac{4\sqrt{x} + 8\sqrt{x} + 3}{4 + 8\sqrt{x} + 3} = \frac{0 + 0 + 3}{4 + 0 + 0} = \frac{3}{4} \]

#2.) Sketch the curve of \( y = \frac{x}{x-1} \)

Step #1.) Domain: All reals with \( x \neq 1 \).

Step #2.) Intercepts:

- **X-intercept**
  
  Let \( y = 0 \). \( 0 = \frac{x}{x-1} \)
  
  \( 0 = x \)
  
  **X-int. @ (1, 0)**

- **Y-intercept**
  
  Let \( x = 0 \)
  
  \( y = \frac{0}{0-1} = \frac{0}{-1} = 0 \)
  
  **Y-int. @ (0, 0)**

Step #3.) Symmetry.

Plug in \((-x)\) for \((x)\)

\[ y = \frac{(-x)}{(-x)-1} = \frac{-x}{-x+1} = \frac{x}{x+1} \]

Since \( \frac{x}{x+1} \neq \frac{x}{x-1} \) then the function is not even.

Since \( \frac{x}{x+1} \neq \frac{-x}{x-1} \) then the function is not odd.
Step #4.1: Asymptotes:

**Horizontal Asymptotes:**

\[
\lim_{x \to \infty} \frac{x}{x-1} = \lim_{x \to \infty} \frac{1}{1} = 1
\]

\[
\lim_{x \to \infty} \frac{x}{x-1} = \lim_{x \to \infty} \frac{1}{1} = 1
\]

**Horizontal Asymptote @ y = 1**

**Vertical Asymptotes:**

This will occur when the denominator is 0.

So when:

\[x-1 = 0\]

\[x = 1\]

Vertical asymptote at x = 1

Step #5.1: Critical numbers and intervals of increase/decrease:

\[y = \frac{x}{x-1}\]

\[y' = \frac{(x-1)\frac{1}{x} - [x(1)]}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2}\]

To find crit. #s we look at where y' is 0 or undefined. Since y' is never 0, we are just concerned where it is undefined.

\[(x-1)^2 = 0\]

\[x = 1\] is a critical #.
Next, we look for intervals of increase/decrease.

Sign of $y'$

\[
\begin{align*}
\text{Choose } x = 0 \text{ for the interval } (-\infty, 1) \\
\frac{dy}{dx} \bigg|_{x = 0} &= \frac{-1}{(0-1)^2} = \frac{-1}{1} = -1 < 0 \\
\text{is decreasing on } (-\infty, 1)
\end{align*}
\]

\[
\begin{align*}
\text{Choose } x = 2 \text{ for the interval } (1, \infty) \\
\frac{dy}{dx} \bigg|_{x = 2} &= \frac{-1}{(2-1)^2} = \frac{-1}{1} = -1 < 0 \\
\text{is decreasing on } (1, \infty) \\
\text{So } y \text{ is decreasing on its entire domain.}
\end{align*}
\]

\boxed{\text{Step #6.}} \quad \text{Local Extrema:} \quad \text{(Since the function never changes from increasing to decreasing or vice versa, there is no local extrema.)}

\boxed{\text{Step #7.}} \quad \text{Concavity and Points of Inflection:} \quad \frac{d^2y}{dx^2} = \frac{-1}{(x-1)^2} = -(x-1)^{-2}

\[
\begin{align*}
\frac{d^2y}{dx^2} &= 2(x-2)^{-3}(1) = \frac{2}{(x-1)^3} \\
\text{There is a possible point of inflection at } x = 1, \text{ since } y'' \text{ is undefined at } x = 1.
\end{align*}
\]
Choose \( x = 0 \) for interval \((- \infty, 1)\) \( \begin{cases} 
\frac{y''}{x = 0} = \frac{2}{(0-1)^2} = \frac{2}{1} = -2 < 0 
\end{cases} \) So \( y \) is concave down on \((- \infty, 1)\).

Choose \( x = 2 \) for interval \((1, \infty)\) \( \begin{cases} 
\frac{y''}{x = 2} = \frac{2}{(2-1)^2} = \frac{2}{1} = 2 > 0 
\end{cases} \) So \( y \) is concave up on \((1, \infty)\).

Since concavity change from concave down to concave up at \( x = 1 \) (vertical asymptote) then \( x = 1 \) is a point of inflection.

Finally, since we don’t have any points for the interval \((2, \infty)\), it is helpful to plot at least one to increase the accuracy of our graph. So \( y' \) \( \begin{cases} 
\frac{y'}{x = 2} = \frac{2}{2-1} = \frac{2}{1} = 2 
\end{cases} \)

Now we have a point at \((2, 2)\).

Sketched Graph of \( y = \frac{x}{x-1} \)