MA 113 — Calculus I Spring 2009
Bypass Exam

Answer all of questions 1–6 and choose two of questions 7–9 to answer. Please indicate which of problems 7–9 is not to be graded by crossing through its number on the table below. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. All other electronic devices including pagers and cell phones should be in the off position for the duration of the exam. Please:

1. **clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may not receive credit).**
2. **give exact answers, rather than decimal approximations to the answer**

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: ______________________________

Question	Score	Total
1		12
2		12
3		12
4		12
5		12
6		12
7		12
8		12
9		12
Free	4	4
		100

(1) Find the following limits. Justify your steps in finding each limit.

(a) $\lim_{t \to 0} \arctan(e^t)$

$= \arctan(e^0)$

$= \arctan(1)$

$= \frac{\pi}{4}$

(b) $\lim_{t \to 0} \frac{t^2}{1 - \cos(t)}$

$= \lim_{t \to 0} \frac{(t^2)'}{(1 - \cos t)'}$

$= \lim_{t \to 0} \frac{2t}{\sin t}$

$= \lim_{t \to 0} \frac{(2t)'}{(\sin t)'}$

$= \lim_{t \to 0} \frac{2}{\cos t} = 2$

(c) $\lim_{x \to \infty} x^{-1/2} \ln x$

$= \lim_{x \to \infty} \frac{\ln x}{x^{\frac{1}{2}}}$

$= \lim_{x \to \infty} \frac{(\ln x)'}{(x^{\frac{1}{2}})'}$

$= \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2} x^{-\frac{1}{2}}}$

$= \lim_{x \to \infty} 2 \cdot \frac{1}{x} \cdot x^{\frac{1}{2}} = 0$

(a) $\frac{\pi}{4}$

(b) 2

(c) 0

(2) Given the function f defined for all x by

$$f(x) = \begin{cases} 3x \text{ for } x \leq 1 \\ Ax^3 + B \text{ for } 1 < x \leq 2 \\ 10 \text{ for } x > 2. \end{cases}$$

(a) Find A, B so that f is continuous for all x.

(b) Determine all points x at which f is not differentiable where A, B are the numbers you found in (a). Indicate your reasoning.

(a) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) \iff 3 = A\cdot 1^3 + B.$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) \iff 10 = A\cdot 2^3 + B.$

$\Rightarrow \begin{cases} A+B=3 \\ 8A+B=10 \end{cases}$ So $\begin{cases} A=1 \\ B=2 \end{cases}$

(b) Note that $\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \frac{3(1+h)-3}{h} = 3$

$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \frac{(1+h)^3+2-3}{h} = 3$

So $f(x)$ is differetiable at $x=1$.

$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \frac{(2+h)^3+2-2^3-2}{h} = 12,$

$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \frac{10-2^3-2}{h} = 0$

As $\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$, $f(x)$ is not differentiable at $x=2$.

Since $f(x)=3x$ on $(-\infty,1)$, then $f(x)$ is differetiable on $(-\infty,1)$.
Similarly, $f(x)$ is also differetiable on $(1,2)$, $(2,+\infty)$.

(a) $A =$ 1 $B =$ 2

(b) Point(points) where f is not differentiable $x=2$

(3) Find the following derivatives. **Show your work!**

(a) $g'(x)$ when $g(x) = x^2 \sec(2x)$.

$$g'(x) = 2x \cdot \sec(2x) + x^2 \cdot \sec'(2x)$$
$$= 2x \cdot \sec(2x) + x^2 \cdot 2\sin 2x \cdot \sec^2 2x$$

(b) dV/dt at $t = \pi/2$ when $V(x) = x^3 + 2x$ and $x = \cos t$.

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = V'(x) \cdot x'(t)$$
$$= (3x^2+2)(-\sin t)$$
$$= (3\cos^2 t + 2)(-\sin t)$$
$$= -3\cos^2 t \sin t - 2\sin t$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{t=\frac{\pi}{2}} = -3\cos^2\frac{\pi}{2} \cdot \sin\frac{\pi}{2} - 2\sin\frac{\pi}{2} = -2$$

(c) $f^{(3)}(x)$ when $f(x) = \dfrac{1}{1-x}$.

$$f'(x) = (-1) \cdot (1-x)^{-2} \cdot (-1) = \frac{1}{(1-x)^2}$$
$$f''(x) = (-2)(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$$
$$f^{(3)}(x) = 2 \cdot (-3)(1-x)^{-4}(-1) = \frac{6}{(1-x)^4}$$

(a) $g'(x) =$ $2x \cdot \sec(2x) + 2x^2 \sin 2x \cdot \sec^2 2x$

(b) dV/dt at $t = \pi/2$ -2

(c) $f^{(3)}(x) =$ $6/(1-x)^4$

(4) The velocity of a particle at time t is given by $v(t) = 8t^3 + 2t + 1$ meters/second.

(a) Find the acceleration, $a(t)$, of the particle at time t.

$$a(t) = v'(t)$$
$$= 8 \cdot 3t^2 + 2$$
$$= 24t^2 + 2$$

(b) If the particle starts at the origin find its position, $s(t)$, at time t.

$$s(t) = \int_0^t (8x^3 + 2x + 1)\,dx$$
$$= (2x^4 + x^2 + x)\Big|_0^t$$
$$= 2t^4 + t^2 + t$$

(a) $a(t) =$ $24t^2+2$ meters/ (second)^2

(b) $s(t) =$ $2t^4+t^2+t$ meters

(5) Given $f(x) = x^2 \ln x$, $x \in (0, 1]$.

(a) Find the critical points of f in (0,1).

(b) Determine whether f has a local maximum or minimum at each critical point. Justify your answer by showing the first or second derivative test is satisfied at each critical point.

(c) Does f have an absolute minimum on (0,1)? Again, indicate your reasoning.

(a) $f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$
$= 2x \cdot \ln x + x$

Let $f'(x) = 0$. Then $2x \ln x + x = 0$.

$x(2\ln x + 1) = 0$

$\Rightarrow x = 0$ or $x = e^{-\frac{1}{2}}$

So the critical point of f in $(0,1)$ is $e^{-\frac{1}{2}}$.

(b) Note that $f''(x) = 2\ln x + 2 + 1$
$= 2\ln x + 3$.

So $f''(e^{-\frac{1}{2}}) = 2 \cdot (-\frac{1}{2}) + 3 = 2 > 0$.

Thus $f(x)$ has a local minimum at $x = e^{-\frac{1}{2}}$

$f(x)$ does not have a local maximum as $x = e^{-\frac{1}{2}}$ is the only critical point of f on $(0,1)$.

(c) As $f'(x) = x(2\ln x + 1) \begin{cases} < 0, & \text{if } x \in (0, e^{-\frac{1}{2}}) \\ > 0, & \text{if } x \in (e^{-\frac{1}{2}}, 1) \end{cases}$

then $f(x)$ is decreasing on $(0, e^{-\frac{1}{2}})$ and increasing on $(e^{-\frac{1}{2}}, 1)$. So $f(x)$ has an absolute minimum at $e^{\frac{1}{2}}$

(a) critical point(points) at $x =$ $e^{-\frac{1}{2}}$

(b) $f(x)$ has a local minimum at $x = e^{-\frac{1}{2}}$ and f has no local max.

(c) $f(x)$ has a absolute Minimum on $(0,1)$.

(6) Find the following integrals. You must show your work to receive credit.

(a) $\int_0^1 t\, e^{t^2}\, dt$

$$= \frac{1}{2}\int_0^1 e^{t^2} dt^2$$

$$= \frac{1}{2} e^{t^2}\Big|_{t=0}^{t=1}$$

$$= \frac{1}{2}(e-1)$$

(b) $\int \frac{1}{\sqrt{1-4x^2}}\, dx$

Let $x = \frac{1}{2}\sin t$. Then $\frac{1}{\sqrt{1-4x^2}} = \frac{1}{\sqrt{1-\sin^2 t}} = \frac{1}{\cos t}$

$$dx = \frac{1}{2}\cos t\, dt.$$

So $\int \frac{1}{\sqrt{1-4x^2}}\, dx = \int \frac{1}{\cos t}\cdot\frac{1}{2}\cdot\cos t\, dt$

$$= \frac{1}{2}t + c$$

$$= \frac{1}{2}\arcsin 2x + c$$

(c) $\int \frac{x}{(x^2+3)^2}\, dx$

$$= \frac{1}{2}\int \frac{dx^2}{(x^2+3)^2}.$$

Let $t = x^2+3$. Then

$$\frac{1}{2}\int \frac{dx^2}{(x^2+3)^2} = \frac{1}{2}\int \frac{dt}{t^2}$$

$$= -\frac{1}{2}\frac{1}{t} + c$$

$$= -\frac{1}{2(x^2+3)} + c$$

(a) $\frac{1}{2}(e-1)$

(b) $\frac{1}{2}\arcsin 2x + c$

(c) $-\frac{1}{2(x^2+3)} + c$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(7) (a) Given f continuous on $[a,b]$. State both versions of the fundamental theorem of calculus.

(1) Let $F(x)$ be the function defined by $F(x) = \int_a^x f(x)dx$ for all $x \in [a,b]$. Then $F(x)$ is continuous on $[a,b]$ and differentiable on (a,b), and

$F'(x) = f(x)$, $\forall x \in (a,b)$.

(2) Let $g(x)$ be an antiderivative of f in $[a,b]$, then $\int_a^b f(x)dx = g(b) - g(a)$.

(b) Illustrate your answer in (a) by finding the derivative of $F(x) = \int_1^x \sin(e^t)\, dt$ at $x = 1$.

Let $f(x) = \sin e^x$, Then $F'(x) = f(x)$ Since $F(x) = \int_1^x f(t)dt$. So $F'(x) = \sin e^x$.

$F'(1) = \sin e^1 = \sin e$

(c) Use the fundamental theorem of calculus and your knowledge of the definite integral to find $\lim_{n\to\infty}\left[(4/n)\sum_{i=1}^{n}(1+i/n)^3\right]$.

Let $f(x) = 4(1+x)^3$, $x \in [0,1]$

Then $\lim_{n\to\infty}\left[\frac{4}{n}\sum_{i=1}^{n}(1+\frac{i}{n})^3\right] = \int_0^1 4(1+x)^3 dx$

$= (1+x)^4 \Big|_{x=0}^{x=1}$

$= 2^4 - 1^4 = 15$

(b) $F'(1) =$ $\sin e$

(c) limit $=$ 15

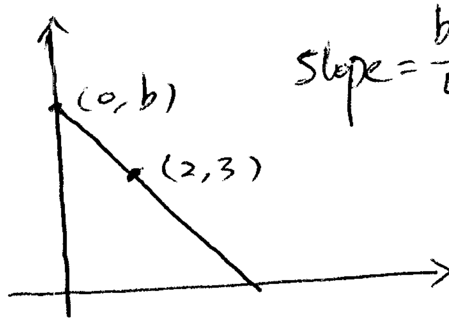
(8) Given a right triangle with one leg on the positive x axis and one leg on the positive y axis. Suppose also the hypotenuse contains (2,3).

(a) Among all such right triangles find the triangle with minimum area.

(b) What is the minimum area?

(a) Assume that the intersection of the hypotenuse and y-axis is $(0, b)$. Then the hypotenuse is on the line $y = \frac{b-3}{-2}x + b$.

Note that $y = \frac{b-3}{-2}x + b$ has x-intercept $\left(\frac{2b}{b-3}, 0\right)$.



$\text{Slope} = \frac{b-3}{0-2}$.

So the hypotenuse intersects x-axis at $\left(\frac{2b}{b-3}, 0\right)$.

$\Rightarrow$ The area of the triangle is

$$A(b) = \frac{1}{2} \cdot b \, \frac{2b}{b-3} = \frac{b^2}{b-3}.$$

$$A'(b) = \frac{2b(b-3) - b^2}{(b-3)^2}. \quad \text{Let } A'(b) = 0.$$

$$\Rightarrow 2b(b-3) - b^2 = 0$$

$$(b^2 - 6b) = 0$$

$$b(b-6) = 0. \quad \text{So } b = 6.$$

$$A''(b) = \frac{2}{b-3} - \frac{4b}{(b-3)^2} + \frac{2b^2}{(b-3)^2}. \quad \text{So } A''(6) = \frac{2}{3} - \frac{24}{9} + \frac{72}{27} = \frac{2}{3} > 0$$

So when $b = 6$, the triangle has minimum area.

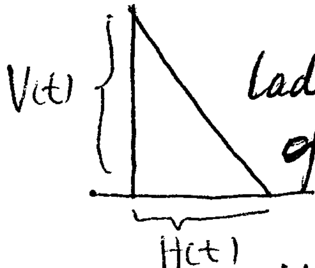
(b) $A(6) = \frac{1}{2} 6 \cdot \frac{12}{3} = 12$

(a) Leg on x axis = 4 Leg on y axis = 6

(b) Area = 12

(9) Initially a 30 foot ladder leans against a vertical wall. The top of the ladder is 25 feet above the floor. At time $t = 0$ the top of the ladder begins sliding down the wall at a rate of 2 feet per second.

(a) How fast is the bottom (or base) of the ladder sliding along the floor 5 seconds later.



Assume that at time t, the height of the ladder is V(t) and the distance from the bottom of the ladder to the wall is H(t).

$$H(t) = \sqrt{30^2 - V^2(t)}$$

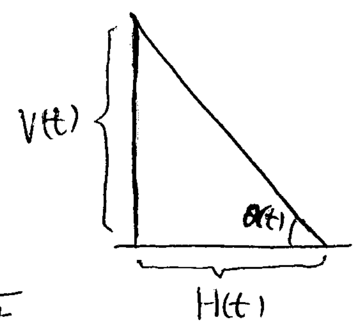
$$H'(t) = \frac{1}{2}(30^2 - V^2(t))^{-\frac{1}{2}}(-2)\,V(t) \cdot V'(t)$$

$$V(5) = 25 - 2\cdot 5 = 15, \quad V'(5) = -2$$

$$\text{So } H'(5) = \frac{1}{2}(900 - 15^2)^{-\frac{1}{2}}(-2)\cdot 15(-2)$$

$$= 30/\sqrt{675} = 1.1547 \text{ ft/s}$$

(b) Let $\theta = \theta(t)$ be the angle that the ladder makes with the floor at time t seconds. Find the rate at which θ is changing with respect to time when $t = 5$.



$$\tan\theta(t) = \frac{V(t)}{H(t)}$$

$$\tan'\theta(t) = \left(\frac{V(t)}{H(t)}\right)'$$

$$\Rightarrow \frac{1}{\cos^2\theta(t)} \cdot \theta'(t) = \frac{H(t)V'(t) - H'(t)V(t)}{H^2(t)}$$

$$V(5) = 15, \quad V'(5) = -2, \quad H(5) = \sqrt{675}$$

$$H'(5) = \frac{30}{\sqrt{675}}, \quad \cos\theta(5) = \frac{H(5)}{30} = \frac{\sqrt{675}}{30}$$

$$\Rightarrow \frac{1}{\cos^2\theta(5)} \cdot \theta'(5) = \frac{H(5)V'(5) - H'(5)V(5)}{H^2(5)}$$

$$\frac{900}{675}\theta'(5) = \frac{\sqrt{675}\cdot(-2) - \frac{30}{\sqrt{675}}\cdot 15}{675}$$

$$\frac{900}{675}\theta'(5) = -0.10264$$

$$\theta'(5) = -0.07698.$$

So when $t=5$, θ is changing at the rate of -0.07698

(a) 1.1547 feet/second

(b) -0.07698 radians/second