MA 113 - Calculus I | Fall 2007
EXAM 1 | September 18, 2007

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please:

1. check answers when possible,
2. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
3. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question. If you use the back of a sheet, please indicate this by the question.

You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name ______________________

Section ___________

Last four digits of student identification number ___________

Question	Score	Total
1		10
2		9
3		9
4		7
5		9
6		9
7		7
8		9
9		14
10		14
11		14
Free	3	3
		100

1. Consider the rational function

$$f(x) = \frac{x-2}{x^2-5x+6}.$$

(a) Determine each of the following limits if it exists.

(i) $\lim_{x \to 1} f(x)$

(ii) $\lim_{x \to 2^-} f(x)$

(iii) $\lim_{x \to 3^+} f(x)$

(b) Which of the lines $x = 1$, $x = 2$, $x = 3$ are vertical asymptotes of the function f?

(a) Before computing a limit we may simplify.
If $x \neq 2, 3$ then $f(x) = \frac{x-2}{x^2-5x+6} = \frac{x-2}{(x-2)(x-3)} = \frac{1}{x-3}$.

(i) $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{x-3} = \frac{1}{1-3} = -\frac{1}{2}$

(ii) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{x-3} = \frac{1}{2-3} = -1$

(iii) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{1}{x-3} = \frac{1}{+0} = \infty$

(b) A vertical asymptote is a line $x = c$ where either of the one-sided limits is infinity or negative infinity

$x = 3$ is a V. A.

(i) $\lim_{x \to 1} f(x) =$ $-\frac{1}{2}$ (ii) $\lim_{x \to 2^-} f(x) =$ -1 (iii) $\lim_{x \to 3^+} f(x) =$ DNE

(b) Vertical asymptotes are $x = 3$

2. Consider the function $f(x) = (\sqrt{x+2} - 3)^{100}$.

(a) Determine the domain of f.

(b) Find all numbers a so that the function f is continuous at a.

(c) Determine $\lim_{x \to 2} f(x)$.

(a) All operations are defined for all real numbers except $\sqrt{x+2}$ which is defined for $x+2 \geq 0$ i.e. $x \geq -2$.

(b) $f(x)$ is the composition of the following functions:

$y_1 = x + 2$ (polynomial)

$y_2 = \sqrt{y_1}$ for $y_1 \geq 0$ (root)

$y_3 = y_2 - 3$ (polynomial)

$y_4 = y_3^{100}$ (power)

Since a composition of continuous functions is continuous, it follows that f is continuous at all $a \geq -2$ (at $a = -2$, f is continuous from the right)

(c) $\lim_{x \to 2} f(x) = f(2) = (\sqrt{2+2} - 3)^{100} = 1^{100} = 1.$

since the function is continuous at $x = 2$.

(a) The domain of f is $[-2, \infty)$

(b) f is continuous at $[-2, \infty)$

(c) $\lim_{x \to 2} f(x) =$ 1

3. Let f and g be two functions such that the following limits exist:

$$\lim_{x\to 2} g(x) = 6 \quad \text{and} \quad \lim_{x\to 2}[xf(x) - 3g(x)] = 2.$$

Show that the limit

$$\lim_{x\to 2} f(x)$$

exists and calculate its value.

Let $x f(x) - 3g(x) = u(x)$ and solve for $f(x)$:

$$x f(x) = u(x) + 3g(x)$$

$$f(x) = \frac{u(x) + 3g(x)}{x} \quad \text{for } x \text{ near } 2.$$

By limit laws, $\lim_{x\to 2} f(x)$ exists and equals

$$\lim_{x\to 2} \frac{u(x) + 3g(x)}{x} = \frac{\lim_{x\to 2} u(x) + 3\lim_{x\to 2} g(x)}{\lim_{x\to 2} x}$$

$$= \frac{2 + 3\cdot 6}{2}$$

$$= 10.$$

$\lim_{x\to 2} f(x) =$ 10

4. Let f be a function such that, for all numbers x,

$$2x^3 + 3 \le f(x) \le x^6 + 4.$$

Show that $\lim_{x\to 1} f(x)$ exists and find its value. As always, justify your answer.

$$\lim_{x\to 1} (2x^3+3) = 2(1)^3 + 3 = 5$$

$$\lim_{x\to 1} (x^6+4) = (1)^6 + 4 = 5$$

Since the two limits exist and have the same value, $\lim_{x\to 1} f(x)$ exists and equals 5 by the squeeze rule.

$\lim_{x\to 1} f(x) =$ 5

5. Let c be a number and consider the function

$$f(x) = \begin{cases} cx^2 - 3 & \text{for } x < 2 \\ x - c & \text{for } x > 2 \end{cases}$$

(a) Find all numbers c such that the limit $\lim_{x \to 2} f(x)$ exists.

(b) Is there a number c such that f is continuous at $x = 2$? As always, justify your answer.

(a) $\lim_{x \to 2^-} f(x) \overset{x<2}{=} \lim_{x \to 2^-} (cx^2 - 3) = 4c - 3$

$\lim_{x \to 2^+} f(x) \underset{x>2}{=} \lim_{x \to 2^+} (x - c) = 2 - c$

$\lim_{x \to 2} f(x)$ exists iff $4c - 3 = 2 - c$

iff $5c = 5$

iff $c = 1$

(b) f is continuous at $x = 2$ if f is defined at $x = 2$, $\lim_{x \to 2} f(x)$ exists, and

$\lim_{x \to 2} f(x) = f(2)$

Since f is not defined at $x = 2$, f is not continuous at $x = 2$.

So that there is no such number c.

(a) $c =$ 1 (b) yes / no (*circle the correct answer*)

6. Using the definition, find the equation of the tangent line to the graph of the function $f(x) = \sqrt{x+1}$ at $x = 3$. Write your result in the form $y = mx + c$.

The slope of the tangent line to $y = \sqrt{x+1}$ at $x=3$ is

$$m = \lim_{x \to 3} \frac{\sqrt{x+1} - \sqrt{3+1}}{x-3}$$

$$= \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x-3} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2}$$

$$= \lim_{x \to 3} \frac{\overset{1}{\cancel{x+1-4}}}{\cancel{(x-3)}(\sqrt{x+1} + 2)}$$

$$= \frac{1}{\sqrt{3+1} + 2} \quad \text{(by continuity)}$$

$$= \frac{1}{4}$$

The equation of the tangent line is the equation of a line with slope $m = \frac{1}{4}$ and passing thru the point $x = 3$, $y = \sqrt{3+1} = 2$.

i.e. $y - 2 = \frac{1}{4}(x-3)$

$$y = \frac{1}{4}x - \frac{3}{4} + 2$$

$$y = \frac{1}{4}x + \frac{5}{4}$$

The equation of the tangent is $y = \frac{1}{4}x + \frac{5}{4}$

7. An apple drops from the top of a tree which is 20 meter tall. Assuming that the apple falls $s = 4.9t^2$ meters after t seconds, find:

(a) the distance the apple traveled after 1 second,

(b) the average velocity between 1 and 2 seconds.

(a) The distance traveled after 1 second is

$$s(1) = 4.9(1)^2 = 4.9 \text{ meters}$$

This distance is less than 20 meters.

(b) The average velocity between 1 and 2 seconds is

$$\frac{s(2)-s(1)}{2-1} = \frac{4.9(2)^2 - 4.9}{1}$$

$$= 4.9(4-1)$$

$$= 4.9(3)$$

$$= 14.7 \text{ meters/second}$$

Observe that $s(2) = 4.9(2)^2 = 4.9(4) = 19.6$ meters is still less than 20 meters.

(a) The distance is 4.9 m (b) The average velocity is 14.7 m/s

8. Consider the function $f(x) = (x+1)^2$.

(a) Determine the slope m_a of the secant line through the points $(1, f(1))$ and $(a, f(a))$ with $a \neq 1$ (as a function of a). As always, simplify your answer.

(b) Find $\lim_{a \to 1} m_a$ if it exists.

(a) $$m_a = \frac{f(a) - f(1)}{a-1} = \frac{(a+1)^2 - (1+1)^2}{a-1}$$

$$= \frac{(a+1-2)(a+1+2)}{a-1}$$

$$= a+3.$$

(b) $$\lim_{a \to 1} m_a = \lim_{a \to 1} (a+3) = 1+3 = 4$$

(a) $m_a =$ $a+3$ (b) $\lim_{a \to 1} m_a =$ 4

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

9. (a) State the principle of mathematical induction.

(b) Prove by mathematical induction that, for all integers $n \geq 1$, the following equality is true:

$$\sum_{k=1}^{n}(2k) = n^2 + n.$$

(a) If a statement depending on an integer $n \geq 1$ is true for $n = 1$ and from the assumption that the statement is true for $n = N$ it follows that the statement is true for $n = N + 1$ where N is any integer ≥ 1, then the statement is true for all integers $n \geq 1$.

(b) For $n = 1$, the equality is true since

$$\sum_{k=1}^{1}(2k) = 2(1) = 2$$

and $(1)^2 + 1 = 1 + 1 = 2$.

From the assumption that the equality is true for $n = N$, it follows that

$$\sum_{k=1}^{N+1}(2k) = \sum_{k=1}^{N}(2k) + 2(N+1)$$
$$= N^2 + N + 2N + 2$$
$$= N^2 + 3N + 2$$

Since $(N+1)^2 + (N+1) = N^2 + 2N + 1 + N + 1 = N^2 + 3N + 2$, it follows that the equation is true for $n = N + 1$. So that the equation holds for all integers $n \geq 1$ by the principle of mathematical induction.

10. (a) State the Intermediate Value Theorem.

(b) Explain why and how you can use this theorem to show that the equation

$$x^5 - 3x^4 + 1 = 0$$

has a root between 0 and 1.

(a) If a function $y = f(x)$ is continuous on a closed interval $[a, b]$ then it takes all the intermediate values, i.e. if $f(a) < N < f(b)$ or $f(a) > N > f(b)$ then there is c in (a, b) such that $N = f(c)$.

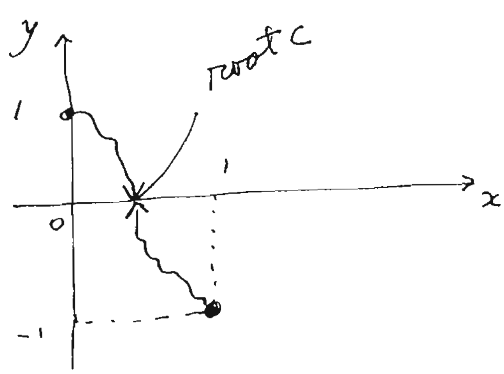
(b) Let $f(x) = x^5 - 3x^4 + 1$. As a polynomial function, f is continuous everywhere.

$$f(0) = (0)^5 - 3(0)^4 + 1 = 1 > 0$$

$$f(1) = (1)^5 - 3(1)^4 + 1 = -1 < 0$$

By applying the theorem to the function f, the closed interval $[0, 1]$ and the intermediate value $f(0) > 0 > f(1)$, it follows that there is c in $(0, 1)$ such that $0 = f(c)$.

This means that the equation has a root c between 0 and 1.



11. (a) State the definition of the derivative of a function f at a point a.

(b) Consider the function $f(x) = |x - 1|$. Using the definition, decide whether f is differentiable at

(i) $x = 2$,

(ii) $x = 1$.

(a) The derivative of a function $y = f(x)$ at $x = a$ is the limit of the quotient

$$\frac{f(x) - f(a)}{x - a}$$

as x approaches a.

(b) (i)

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{|x-1| - 1}{x - 2} \underset{x>1}{=} \lim_{x \to 2} \frac{x - 1 - 1}{x - 2} = \lim_{x \to 2} \frac{x-2}{x-2} = 1$$

Since the limit exists, f is differentiable at $x = 2$.

(ii)

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x-1|}{x-1} \underset{x>1}{=} \lim_{x \to 1^+} \frac{x-1}{x-1} = 1$$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x-1|}{x-1} \underset{x<1}{=} \lim_{x \to 1^-} \frac{-x+1}{x-1} = -1$$

Since the limit does not exist, f is not differentiable at $x = 1$.