Calculus I
Exam 1

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Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*.
Each question is followed by space to write your answer. Please lay out your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question.
You are to answer three of the last four questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name __________________________

Section _____________

Last four digits of student identification number _____________

Question	Score	Total
1		6
2		6
3		6
4		6
5		8
6		8
7		10
8		6
9		6
Q10		12
Q11		12
Q12		12
Q13		12
Free		2
		100

1. Let $f(x) = 1 - x^2$ and $g(x) = 1/x$.
   Find $f(g(2))$ and $g(f(2))$.
   Find the domain of $f(g(x))$.

$f(g(x)) = 1 - \frac{1}{x^2}$ $\quad f(g(2)) = 1 - \frac{1}{4} = \frac{3}{4}$

$g(f(x)) = \frac{1}{1-x^2}$ $\quad g(f(2)) = \frac{1}{1-4} = -\frac{1}{3}$

$f(g(x))$ is not defined at $x = 0$.

6 total

2 2 2

$f(g(2)) =$ 3/4, $g(f(2)) =$ $-1/3$, Domain of $f(g(x))$ $\mathbb{R} \setminus \{0\}$ or all real numbers except 0.

2. Find the quadratic function $f(x) = ax^2 + bx + c$ which has $f(1) = f(4) = 0$ and $f(2) = 1$.
   Give your answer in the form $ax^2 + bx + c$.

2 $f(x)$ has the form $f(x) = a(x-1)(x-4)$

2 $f(2) = 1 = a(1)(-2) = -2a$ so $a = -\frac{1}{2}$

2 $f(x) = -\frac{1}{2}(x^2 - 5x + 4) = -\frac{1}{2}x^2 + \frac{5}{2}x - 2$

6 total

$-\frac{1}{2}x^2 + \frac{5}{2}x - 2$

3. The distance d (in kilometers) a train travels from a station is given as a function of time t (in hours) by the formula: $d(t) = 2t^2 + 5t$, for $t \geq 0$. Find the average velocity of the train between times $t = 2$ and $t = 4$.

3

$$\text{Average velocity} = \frac{\Delta d}{\Delta t} = \frac{d(4) - d(2)}{4-2} = \frac{52-18}{2}\ \text{km/hr}$$

$$= \frac{34}{2}\ \text{km/hr} \overset{2}{=} 17\ \text{km/hr} \quad 1 \text{ for units}$$

6 total

Average velocity = 17 km/hr

4. Suppose that f is continuous and that we know the following limits:

$$\lim_{x\to 0} f(x) = 4 \quad \lim_{x\to 0} g(x) = 2$$
$$\lim_{x\to 2} f(x) = 1 \quad \lim_{x\to 2} g(x) = 4$$
$$\lim_{x\to 4} f(x) = 3 \quad \lim_{x\to 4} g(x) = 0$$

6 points total

Compute the following limits and give your reasoning: a) $\lim_{x\to 4} f(g(x)) =$ 4

b) $\lim_{x\to 2} (fg)(x) =$ 4

a) f continuous so $\lim_{x\to 4} f(g(x)) = f(\lim_{x\to 4} g(x))$

3

$= f(0) = 4$

Note: $\lim_{x\to 0} f(x) = f(0) = 4$

b) $\lim_{x\to 2} f(x)g(x) = f(2)\cdot\left(\lim_{x\to 2} g(x)\right) = f(2)\cdot 4$

3

$= 1 \cdot 4 = 4$

5. Compute the following two limits giving a clear statement of your reasoning:

(a) 4

$$\lim_{x\to -1} \frac{x^3-1}{x-1} = \underline{\quad 1 \quad}$$

(b) 4

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \underline{\quad 3 \quad}$$

8 pts total

a) $\left.\begin{array}{l}\lim\limits_{x\to -1}(x^3-1) = -2 \\ \lim\limits_{x\to -1}(x-1) = -2\end{array}\right\}$ polynomials

2

2 since the limit of the denominator exists and is not zero, the limit is the quotient of the limits.

b) This time $\lim\limits_{x\to 1}(x^3-1) = 0 = \lim\limits_{x\to 1}(x-1)$ so we

2 see if $(x-1)$ divides (x^3-1).

Division: (For example)

$$\begin{array}{r} x^2+x+1 \\ x-1 \,\overline{)\, x^3-1} \\ \underline{x^3-x^2} \\ x^2-1 \\ \underline{x^2-x} \\ x-1 \end{array}$$

2

so $\lim\limits_{x\to 1} \frac{x^3-1}{x-1} = \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = \lim\limits_{x\to 1}(x^2+x+1)$

8 points

6. Are the following functions continuous at the indicated points? Give your reasoning.

(a) $f(x) = \frac{x+1}{x^3+2x-1}$ at $x = 1$. Yes

4 pts

f is a rational function, Each polynomial is continuous and $\lim_{x\to 1}(x+1) = 2$ $\lim_{x\to 1}(x^3+2x-1) = 2 \neq 0$

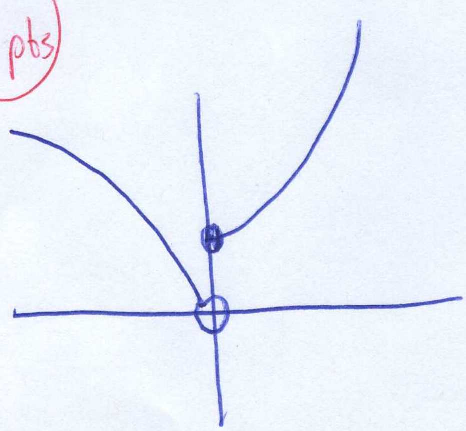
whence $f(1) = 2/2 = 1$, and $\lim_{x\to 1} f(x) = f(1)$.

(b) Is g continuous at $x = 0$ where g is defined by

4 pts

$$g(x) = \begin{cases} x^2+1 & x \geq 0 \\ \sqrt{-x} & x < 0 \end{cases}$$

No



$\lim_{x\to 0^+} g(x) = 1$; $\lim_{x\to 0^-} g(x) = 0$

The left & right limits exist but aren't equal so $\lim_{x\to 0} g(x)$ does not exist and g is not continuous at $x = 0$

7. Find the derivatives of the following functions using the differentiation rules. Simplify your answers.

10 pts

(a) $g(x) = 3x^5 - 2x^3 + 1$

(b) $h(t) = \dfrac{1+t}{2+t^2}$

(c) $f(x) = 5x^{-3/5}$

3

a) $g'(x) = 15x^4 - 6x^2$

4

b) Quotient Rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$u(t) = 1+t \quad u'(t) = 1$

$v(t) = 2+t^2 \quad v'(t) = 2t$

$$h'(t) = \frac{1(2+t^2) - (1+t)(2t)}{(2+t^2)^2} = \frac{-t^2 - 2t + 2}{(2+t^2)^2}$$

3

c) Power law

$$f'(x) = 5 \cdot \left(-\frac{3}{5}\right) x^{-3/5 - 1} = -3x^{-8/5}$$

(6 pts)

8. Consider the following function:

$$g(x) = \begin{cases} -x & x \leq 0 \\ x^2 & 0 < x \end{cases}$$

(a) Sketch the graph of g.

(b) Is g differentiable at $x = 0$? State your reason why or why not.

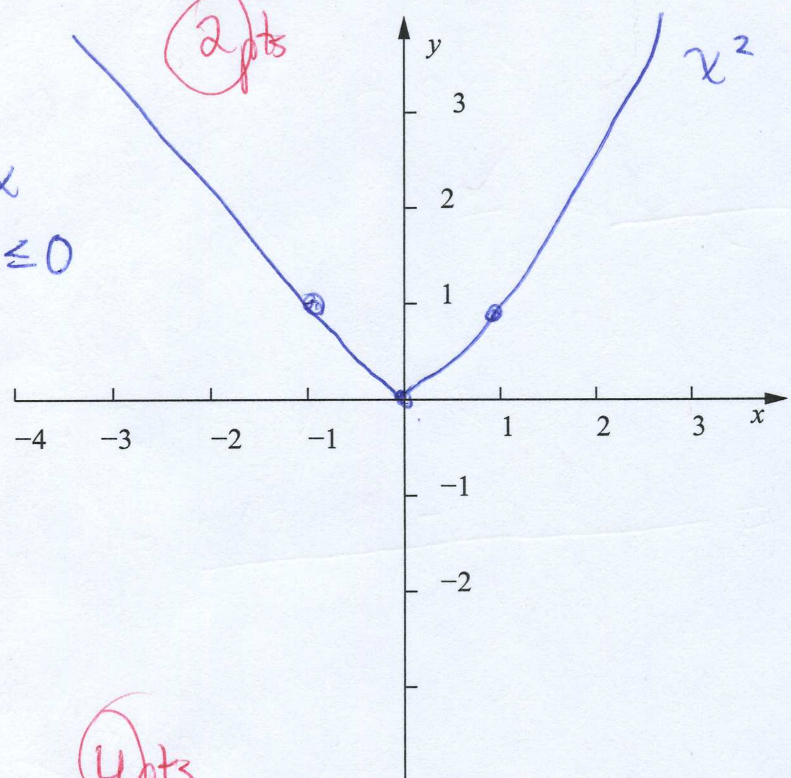
NO

(2 pts)

$x^2 \quad x > 0$

$-x$

$x \leq 0$



(4 pts)

At $x=0$

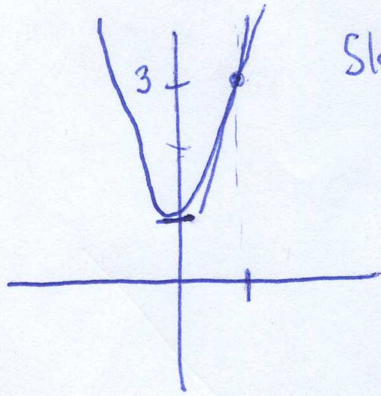
$$\lim_{h \to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^-}\left(\frac{-h}{h}\right) = -1$$

$$\lim_{h \to 0^+} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$$

Since the left and right limits are not equal the limit, and thus the derivative, does not exist at $x=0$.

6 pts

9. Find the equation of the tangent line to the graph of the function $f(x) = 2x^2 + 1$ at $x = 1$. Specify the slope and put the equation of the tangent line in the form $y = mx + b$.



Slope is given by $f'(1)$ 3

$f'(x) = 4x$ $\boxed{f'(1) = 4}$ slope

The tangent line passes through $(1, f(1)) = (1, 3)$ and has slope 4:

$\frac{y-3}{x-1} = 4$ or $y = 4x - 4 + 3$ 3

$$\boxed{y = 4x - 1}$$

4 4x-1

Answer three of the following four questions. Indicate clearly which question is not to be graded by drawing a line through the question number in the table on the front of the exam.

12 pts

10. (a) State the principle of mathematical induction. Use complete sentences.

(b) Use the principle of mathematical induction to prove that

$$\sum_{k=1}^{n} 2k = n^2 + n.$$

a) Given statements $P_1, P_2, \ldots, P_n, \ldots$

Suppose P_1 is true. If we assume P_n is true

4 and we can then prove P_{n+1} is true, it follows

that all P_k are true $k = 1, 2, 3, \ldots$

b) Let P_n be the statement $\sum_{k=1}^{n} 2k = n^2 + n$.

4 $\underline{n=1}$ P_1 $\sum_{k=1}^{1} 2k = 2 = 1+1$ This is true.

$\underline{\text{Induction}}$ Suppose $\sum_{k=1}^{n} 2k = n^2+n$. Now we try

to prove P_{n+1}:

$$\sum_{k=1}^{n+1} 2k = \sum_{k=1}^{n} 2k + 2(n+1) = \underbrace{(n^2+n)}_{\text{by assumption } P_n} + 2n+2$$

4

$$= n^2 + 3n + 2$$

$$= (n+1)^2 + (n+1)$$

verifying P_{n+1}.

12pts

11. (a) Suppose the tangent line to the parabola $y = x^2$ at $x = a$ passes through the point $(x, y) = (2, 3)$. Write an equation that a must satisfy.

    (b) Find all solutions to the equation you wrote in part a).

    (c) Find all tangent lines to the parabola $y = x^2$ which pass through the point $(2, 3)$. Put your answer(s) in the form $y = mx + b$.

a) Note that $(2,3)$ is not on the parabola. The slope of the tangent at $x=a$ is $y'(x) = 2x$ so

4 $m = y'(a) = 2a$. The 2 pts (a, a^2) and $(2,3)$ are on the line of slope $2a$ so:

$$\frac{a^2-3}{a-2} = 2a \quad \text{or} \quad a^2 - 3 = 2a^2 - 4a$$

$$\boxed{0 = a^2 - 4a + 3}$$

b) Solve this quadratic:

4 $a^2 - 4a + 3 = (a-3)(a-1)$ $\boxed{a=1 \text{ and } a=3 \text{ are the roots}}$

c) $a=1$ The tangent passes through $(2,3)$ with slope 2:

2 $\frac{y-3}{x-2} = 2$ or $\boxed{y = 2x - 1}$ also passes through $(1,1)$.

4 $a=3$ The tangent passes through $(2,3)$ with slope 6:

2 $\frac{y-3}{x-2} = 6$ or $\boxed{y = 6x - 9}$

it also passes through $(3,9)$.

12 pts

12. (a) State the definition of the derivative of a function f at a point a. Use complete sentences.

(b) Use the definition of the derivative to find the derivative of $f(x) = \sqrt{1+2x}$

(c) Give the domain of the derivative, f'.

a) The derivative of $f(x)$ at the point $x=a$ is defined as $f'(a) = \lim_{x\to a} \frac{f(a)-f(x)}{a-x}$ or equivalently by

$f'(a) = \lim_{h\to 0} \frac{f(h+a)-f(a)}{h}$. (They need to give one or the other)

4

b) Compute: $f(x+h)-f(x) = \left(\sqrt{1+2(x+h)} - \sqrt{1+2x}\right)\left(\frac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}\right)$

$$= \frac{1+2(x+h)-(1+2x)}{\sqrt{1+2(x+h)}+\sqrt{1+2x}} = \frac{2h}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}$$

4

Take the difference quotient & the limit

$$\boxed{f'(x) = \lim_{h\to 0} \frac{2}{\sqrt{1+2(h+x)}+\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}}.$$

c) The domain of $f'(x)$ is all x such that $1+2x>0$ or $\boxed{x>\frac{-1}{2}}$ (strict inequality)

4

13. (a) State the intermediate value theorem. Use complete sentences. 12pts

(b) Use the intermediate value theorem to find an interval $[a, b]$ so that the equation $x^5 + 2x^3 - 2x^2 + 4 = 0$ has a solution in the open interval (a, b).

a) Suppose f is continuous on $[a,b]$ and that $f(a) \neq f(b)$. If y_0 is any value between $f(a)$ & $f(b)$, then there exists $c \in (a,b)$ so $f(c) = y_0$.

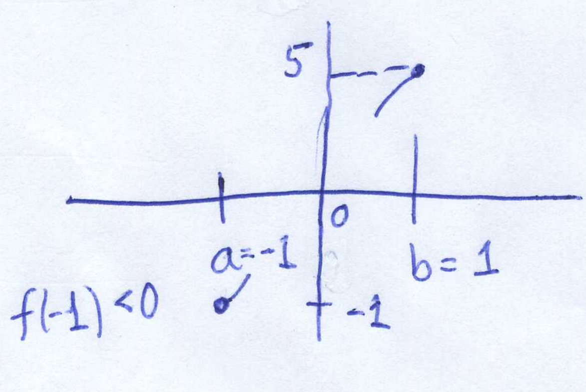
6

b) We want to guess a & b so $f(x) = x^5 + 2x^3 - 2x^2 + 4$ has different signs at a & b. Since f is continuous, the Intermediate Value Theorem implies that there is a point $c \in (a,b)$ so $f(c) = 0$.

6

Guess! $b = 1$ $\quad f(1) = 5 > 0$ Pos 3

$a = -1$ $\quad f(-1) = -1 < 0$ Neg 3



Also $a = -1$
$b = 0$ $\quad f(0) = 4$