MA 113 — Calculus I Spring 2010
Exam 1 February 9, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Key

Section: __________

Last four digits of student identification number: __________

Question	Score	Total
1		8
2		8
3		9
4		8
5		13
6		8
7		11
8		16
9		16
10		16
Free	3	3
		100

(1) (a) Solve the equation $3^{2x+5} = 4$. **Show all steps of the computation and give the exact answer.**

2 Taking $\log_3$ we get $\quad 2x+5 = \log_3 4$

2 $\quad 2x = \log_3 4 - 5$

$$x = \frac{\log_3 4 - 5}{2}$$

(b) Express the quantity

$$\log_2(x^3 - 2) + \frac{1}{3}\log_2(x) - \log_2(5x)$$

as a single logarithm.

The laws of logarithms provide

$$\log_2(x^3-2) + \frac{1}{3}\log_2(x) - \log_2(5x)$$

4 $$= \log_2(x^3-2) + \log_2(x^{\frac{1}{3}}) - \log_2(5x)$$

$$= \log_2\left(\frac{(x^3-2)\sqrt[3]{x}}{5x}\right)$$

(a) $\dfrac{\log_3 4 - 5}{2}$

(b) $\log_2\left(\dfrac{(x^3-2)\sqrt[3]{x}}{5x}\right)$

(2) Consider the functions $f(x) = \sqrt{12 - 2x}$ and $g(x) = x^2 + 2$.

(a) Compute $f(g(1))$ and $g(f(1))$. **Give exact answers.**

① $f(g(1)) = f(3) = \sqrt{12-6} = \sqrt{6}$

① $g(f(1)) = g(\sqrt{10}) = (\sqrt{10})^2 + 2 = 12$

(b) Let h be the composite function $h(x) = (f \circ g)(x)$. Find the domain of h. **As usual, justify your answer by showing your work.**

②
$$\begin{aligned} h(x) = (f \circ g)(x) &= f(g(x)) \\ &= f(x^2+2) \\ &= \sqrt{12 - 2(x^2+2)} \\ &= \sqrt{8 - 2x^2} \end{aligned}$$

② Hence $h(x)$ is defined if and only if $8 - 2x^2 \geq 0$,

① thus $8 \geq 2x^2$,

$4 \geq x^2$,

which means $-2 \leq x \leq 2$.

① Hence the domain is $[-2, 2]$

(a) $f(g(1)) =$ $\sqrt{6}$, $g(f(1)) =$ 12

(b) Domain of h is $[-2, 2]$

(3) Consider the function

$$f(x) = \frac{6x+3}{2x-1}.$$

(a) Find the domain of f.

1 $f(x)$ is defined if and only if $2x-1 \neq 0$, that is $x \neq \frac{1}{2}$.

1 Hence the domain of f is $\{x \mid x \neq \frac{1}{2}\}$, which can also be written as $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$.

(b) Find the inverse function $f^{-1}(x)$ of $f(x)$.

2 Set $y = f(x) = \dfrac{6x+3}{2x-1}$ and solve for x:

3
$$y(2x-1) = 6x+3, \text{ thus}$$
$$2xy - 6x = y+3, \text{ so}$$
$$x(2y-6) = y+3.$$

No deduction if not discussed. It follows that $2y-6 \neq 0$ because otherwise $y=3$ and the left-hand side of the equation is zero whereas the right-hand side is $3+3=6 \neq 0$, a contradiction. Hence, we can divide by $2y-6$ and obtain

1
$$x = \frac{y+3}{2y-6}.$$

1 Hence $f^{-1}(x) = \dfrac{x+3}{2x-6}$.

(a) Domain of f is $\{x \mid x = \frac{1}{2}\}$

(b) $f^{-1}(x) = \dfrac{x+3}{2x-6}$

(4) Let f be a function such that, for all real numbers x near 4,

$$4x - 9 \leq f(x) \leq x^2 - 4x + 7.$$

Argue that $\lim_{x \to 4} f(x)$ exists and find its value. **As usual, justify each step of your work.**

Since polynomial functions are continuous, we get

2 $\lim_{x \to 4} (4x-9) = 4 \cdot 4 - 9 = 7$

2 and $\lim_{x \to 4} (x^2 - 4x + 7) = 16 - 16 + 7 = 7.$

2 Since both limits agree, the Squeeze Theorem gives that $\lim_{x \to 4} f(x)$ exists

2 and $\lim_{x \to 4} f(x) = \underline{\underline{7}}.$

$\lim_{x \to 4} f(x) =$ 7

(5) Use the limit rules to determine each of the following limits if it exists. If a limit does not exist, but is ∞ or $-\infty$, then clearly indicate that.

2

(a) $\lim_{x\to 4} \frac{x^2+\sqrt{x}}{2^x-10} \overset{1}{=} \frac{4^2+\sqrt{4}}{2^4-10} = \frac{18}{6} \overset{1}{=} 3$ because $\frac{x^2+\sqrt{x}}{2^x-10}$ is continuous on its domain.

4

(b) $\lim_{x\to 5} \frac{x^2-4x-5}{x^2-8x+15} \overset{2}{=} \lim_{x\to 5} \frac{(x-5)(x+1)}{(x-5)(x-3)} \overset{1}{=} \lim_{x\to 5} \frac{x+1}{x-3} \overset{1}{=} \frac{5+1}{5-3} = 3$

because 5 is in the domain of the continuous function $f(x) = \frac{x+1}{x-3}$.

3

(c) $\lim_{x\to 4} \frac{1-x}{(2x-8)^2} \overset{1}{=} -\infty$ because $\lim_{x\to 4} (1-x) \overset{1}{=} -3$ and $\lim_{x\to 4} (2x-8)^2 \overset{1}{=} 0^+$ as $(2x-8)^2 \geq 0$.

4

(d) $\lim_{h\to 0} \frac{(h-2)^2-4}{h} \overset{1}{=} \lim_{h\to 0} \frac{h^2-4h+4-4}{h} \overset{1}{=} \lim_{h\to 0} \frac{h^2-4h}{h} = \lim_{h\to 0} \frac{h(h-4)}{h}$

$\overset{1}{=} \lim_{h\to 0} (h-4) = 0-4 \overset{1}{=} -4$ because polynomial functions are continuous.

(a) 3

(b) 3

(c) $-\infty$

(d) -4

(6) Let f and g be two functions such that the following limits exist

$$\lim_{x\to 2} g(x) = 5, \quad \lim_{x\to 2}\left[x^2 f(x) - 3^x g(x)\right] = 15.$$

Use the limit laws to compute the following limits.

3

(a) $\lim_{x\to 2} \frac{x^2+4}{g(x)}.$ $= \frac{\lim_{x\to 2}(x^2+4)}{\lim_{x\to 2} g(x)} = \frac{2^2+4}{5} = \frac{8}{5}$

by limit laws

polynomial functions are continuous

(b) $\lim_{x\to 2} f(x)$. Set $L := \lim_{x\to 2} f(x)$. Then the limit laws provide

3

$$\underbrace{\left(\lim_{x\to 2} x^2\right)}_{=2^2=4}\cdot\underbrace{\left(\lim_{x\to 2} f(x)\right)}_{=L} - \underbrace{\left(\lim_{x\to 2} 3^x\right)}_{=3^2=9}\underbrace{\left(\lim_{x\to 2} g(x)\right)}_{5} = 15,$$

where the limits are obtained by direct substitution because x^2 and 3^x are continuous. Now we solve the equation $4\cdot L - 9\cdot 5 = 15$ for L:

2

$$4L = 60$$

$$L = 15.$$

(a) $\frac{8}{5}$

(b) 15

(7) An apple drops from a tall tree. The apple falls $s(t) = 5t^2$ meters after t seconds. In the following problems include units when stating your answers.

(a) Find the average velocity of the apple over the time interval $1 \le t \le 1.5$.

3

This is $\frac{s(1.5) - s(1)}{1.5 - 1} = \frac{5 \cdot 2.25 - 5}{0.5} = 10 \cdot 1.25 = 12.5 \frac{m}{s}$

1 1 1

(b) Find the average velocity over the time interval $[1, t]$, where $t > 1$. **Simplify your answer.**

5

$\frac{s(t) - s(1)}{t-1} = \frac{5t^2 - 5}{t-1} = 5 \frac{(t+1)(t-1)}{t-1} = 5(t+1) \frac{m}{s}$

1 1 1 2

(c) Use your answer in (b) to find the instantaneous velocity of the apple after 1 second.

3

This is $\underbrace{\lim_{t \to 1} \frac{s(t)-s(1)}{t-1}}_{1} \overset{(b)}{=} \lim_{t \to 1} 5(t+1) = 5(1+1) = 10 \frac{m}{s}$.

1 1

polynomial functions are continuous

(a) Average velocity over [1, 1.5] is $12.5 \frac{m}{s}$

(b) Average velocity over [1, t] is $5(t+1) \frac{m}{s}$

(c) Instantaneous velocity at time $t = 1$ is $10 \frac{m}{s}$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) Define what it means for a function f to be continuous at a. Use complete sentences.

(4) A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

Let

$$f(x) = \begin{cases} cx^2 + 2x - 4, & \text{if } x < 3, \\ 4, & \text{if } x = 3, \\ \frac{12c}{x} + 12, & \text{if } x > 3. \end{cases}$$

For the following problems, always justify your answer!

(b) Find all values for c such that $\lim_{x\to 3} f(x)$ exists.

Since cx^2+2x-4 and $\frac{12c}{x}+12$ are continuous at 3, we get

(2) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (cx^2-2x-4) = 9c+2$

(2) and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (\frac{12c}{x}+12) = 4c+12$.

(2) The two one-sided limits agree iff $9c+2 = 4c+12$ iff $5c=10$ iff $c=2$

(1) Hence $\lim_{x\to 3} f(x)$ exists iff $c=2$.

(c) For which of the values for c found in (b) is the function f continuous at 3?

(3) If f is continuous at 3, then $\lim_{x\to 3} f(x)$ must exist. By (b), this forces $c=2$. However, if $c=2$, then $\lim_{x\to 3} f(x) = 9\cdot 2+2 = 20 \neq 4 = f(3)$. Hence f is never continuous at 3.

(d) Find all values for c such that the function f is continuous at 0.

(2) Since polynomial functions are continuous, we get $\lim_{x\to 0} f(x) = \lim_{x\to 0}(cx^2+2x-4) = -4 = f(0)$. Hence f is continuous at 0 for every number c.

(b) 2 (c) none (d) $\mathbb{R}$

(9) (a) State the Intermediate Value Theorem. Use complete sentences.

7 If a function f is continuous on a closed interval $[a,b]$ and N is a number strictly between $f(a)$ and $f(b)$, then there is a number c in the open interval (a,b) such that $f(c) = N$.

(b) **Explain in detail why and how** you can use this theorem to show that the equation

$$6^x - 3x^2 - 5x = 2$$

has a solution in the interval (1, 2).

2 Consider the function $f(x) = 6^x - 3x^2 - 5x - 2$. Since polynomial and exponential functions are continuous, f is continuous.

2 In particular, f is continuous on $[1,2]$.

2 Now we compute $f(1) = 6 - 3 - 5 - 2 = -4 < 0$

and $f(2) = 36 - 12 - 10 - 2 = 12 > 0$.

2 Hence $N := 0$ is strictly between $f(1)$ and $f(2)$, thus the IVT gives the existence of some c in $(1,2)$ such that

$$0 = f(c) = 6^c - 3c^2 - 5c - 2,$$

so $2 = 6^c - 3c^2 - 5c$.

1 Therefore c is the desired solution.

(10) (a) State the definition of the derivative of a function at a point a. Use complete sentences.

3 The derivative of f at a is $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ (or equivalently $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$),

1 provided the limit exists.

(It is enough to mention one of the two limits.)

Consider now the function

$$f(x) = \frac{1}{2x-7}.$$

(b) Compute the slope of the secant line through the points $(4, f(4))$ and $(5, f(5))$.

2 The slope is $\frac{f(5)-f(4)}{5-4} = \frac{1}{3} - 1 = \underline{\underline{-\frac{2}{3}}}$.

(c) Compute the slope of the secant line through the points $(4, f(4))$ and $(4+h, f(4+h))$, where $h \neq 0$. **Simplify your answer.**

4 The slope is $\frac{f(4+h)-f(4)}{(4+h)-4} = \frac{\frac{1}{2(4+h)-7} - 1}{h} = \frac{1-(1+2h)}{h(1+2h)}$

$$= \frac{-2h}{h(1+2h)} = \underline{\underline{-\frac{2}{1+2h}}}.$$

(d) Use part (c) to compute $f'(4)$.

3 $f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \overset{(c)}{=} \lim_{h\to 0} -\frac{2}{1+2h} = -\frac{2}{1+0} = \underline{\underline{-2}}$

rational functions are continuous on their domain.

(e) Compute the equation of the tangent line to the graph of f at the point $(4, f(4))$. Put your answer in the form $y = mx + b$.

3 By (d), the slope is -2 and the line passes through $(4, f(4)) = (4, 1)$. Hence the equation of the tangent line is $y - 1 = -2(x-4)$, so

$$\underline{\underline{y = -2x + 9}}.$$