Calculus I
Exam 2

Common exam
17 October 2006

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*.
Each question is followed by space to write your answer. Please lay out your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

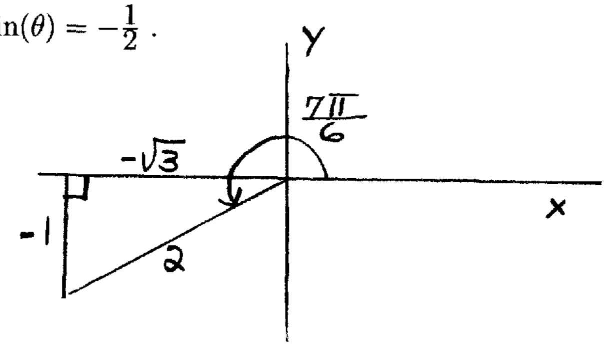
Name Key

Section

Last four digits of student identification number

Question	Score	Total
p. 1/Q1–3		14
p. 2/Q4		14
p. 3/Q5,6		14
p. 4/Q7,8		14
p. 5/Q9		14
p. 6/Q10		14
p. 7/Q11		14
p. 8/Q12		14
Free	2	2
		100

5 1. Find the exact value of the angle θ in radians that satisfies both $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $\sin(\theta) = -\frac{1}{2}$.



$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

+4 $\theta =$ $\frac{7\pi}{6}$

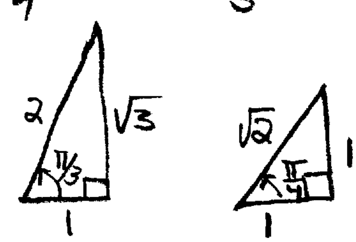
5 2. Using the addition or subtraction formula for the sine function and the equality $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$, obtain the exact value for $\sin(\frac{\pi}{12})$.

+2

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \quad +2$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} \quad +1$$



$\sin(\frac{\pi}{12}) =$ $\frac{\sqrt{6}-\sqrt{2}}{4}$

3. Compute $\lim_{x\to 0} \frac{\sin(3x)}{\sin(x)}$. Your solution should show the limit laws you use.

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(x)} = \lim_{x\to 0} 3\,\frac{\frac{\sin(3x)}{3x}}{\frac{\sin(x)}{x}} \quad (+2) \quad = 3\,\frac{\lim_{x\to 0}\frac{\sin(3x)}{3x}}{\lim_{x\to 0}\frac{\sin(x)}{x}} \quad (+1)$$

$h = 3x$

$$= 3\,\frac{\lim_{h\to 0}\frac{\sin(h)}{h}}{1} \quad (+2) \quad = 3\left(\frac{1}{1}\right) = 3$$

$\lim_{x\to 0} \frac{\sin(3x)}{\sin(x)} =$ 3

14pts

4. Compute the indicated derivatives below showing your steps:

(a) $\frac{d}{dx}\,\frac{1}{x^2+x}$

(b) $\frac{d}{dx}\,\sin^4(x)$

(c) $\frac{d}{dx}\,x\cos\left(\frac{1}{x}\right)$

(d) $\frac{d}{dx}\,\frac{1}{\sqrt{4+3x}}$

a) $\frac{d}{dx}\frac{1}{x^2+x} = -\frac{1}{(x^2+x)^2}\frac{d}{dx}(x^2+x)$

+3 $\quad = -\frac{2x+1}{(x^2+x)^2} \quad x \neq 0, -1$

b) $\frac{d}{dx}\sin^4(x) = 4\sin^3(x)\frac{d}{dx}\sin(x)$

+3 $\quad = 4\sin^3(x)\cos(x)$

c) $\frac{d}{dx}\,x\cos\left(\frac{1}{x}\right) = x\frac{d}{dx}\cos\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\frac{d}{dx}x$

$= x\left[-\sin\left(\frac{1}{x}\right)\frac{d}{dx}\frac{1}{x}\right] + \cos\left(\frac{1}{x}\right)$

+4

$= -x\sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \cos\left(\frac{1}{x}\right)$

$= \frac{1}{x}\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) \quad x \neq 0$

d) $\frac{d}{dx}\frac{1}{\sqrt{4+3x}} = \frac{d}{dx}(4+3x)^{-\frac{1}{2}} = -\frac{1}{2}(4+3x)^{-\frac{3}{2}}\frac{d}{dx}(4+3x)$

$= -\frac{3}{2(4+3x)^{3/2}} \quad x \neq -\frac{4}{3}$

+4

Note Don't need to give domains

(a) $\frac{d}{dx}\,\frac{1}{x^2+x} =$ $-\frac{2x+1}{(x^2+x)^2}$

(b) $\frac{d}{dx}\,\sin^4 x =$ $4\sin^3(x)\cos(x)$

(c) $\frac{d}{dx}\,x\cos\left(\frac{1}{x}\right) =$ $\frac{1}{x}\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$

(d) $\frac{d}{dx}\,\frac{1}{\sqrt{4+3x}} =$ $\frac{-3}{2(4+3x)^{3/2}}$ $= -\frac{3}{2}(4+3x)^{-3/2}$

either

5. Find the equation of the line tangent to the graph of $xy^4 + y = 2x - 2$ at the point $(3, 1)$. Put your answer in the form $y = mx + b$.

$$\frac{d}{dx}(xy^4 + y) = \frac{d}{dx}(2x - 2)$$

$$x\frac{d}{dx}y^4 + y^4\frac{d}{dx}x + \frac{dy}{dx} = 2$$

3 $$x\left(4y^3\frac{dy}{dx}\right) + y^4 + \frac{dy}{dx} = 2$$

$$(4xy^3 + 1)\frac{dy}{dx} = 2 - y^4$$

$$\frac{dy}{dx} = \frac{2 - y^4}{4xy^3 + 1}$$

At $(3,1)$, +2

$$\frac{dy}{dx} = \frac{2-1}{4(3)(1)+1} = \frac{1}{13}$$

Tangent line

$$y - 1 = \frac{1}{13}(x - 3)$$

$$y = 1 + \frac{x}{13} - \frac{3}{13}$$

$$= \frac{x}{13} + \frac{10}{13}$$

+2

$y =$ $\frac{x}{13} + \frac{10}{13}$

6. A projectile is fired straight upward out the window of a building. The height of the projectile in meters above the ground t seconds afterward is given by $h(t) = 80 + 30t - 5t^2$.

(a) Find the acceleration a of the projectile at its highest point.

(b) Find the velocity v of the projectile when it hits the ground.

$$v = \frac{dh}{dt} = \frac{d}{dt}(80 + 30t - 5t^2)$$

$$= 30 - 10t \quad +2$$

$$a = \frac{dv}{dt} = -10 \text{ m/sec}^2 \quad +2$$

When projectile hits ground, $h(t) = 0$, i.e., $80 + 30t - 5t^2 = 0$,

so $5(t^2 - 6t - 16) = 5(t-8)(t+2) = 0$.

Thus $t = 8$ sec so $v = 30 - 10(8) = -50$ m/sec.

+2 +1

(a) $a =$ -10 m/sec^2, (b) $v =$ -50 m/sec

7 pts

7. Suppose $f(1) = 2$, $f'(1) = 3$, $f''(1) = 5$ and define a function h by $h(x) = f(x^2)$. Find the derivatives $h'(1)$ and $h''(1)$. By the chain rule,

+2 $h'(x) = f'(x^2)\frac{d}{dx}x^2 = 2xf'(x^2)$ and

$h''(x) = \frac{d}{dx}2xf'(x^2) = 2f'(x^2)\frac{d}{dx}x + 2x\frac{d}{dx}f'(x^2)$

$= 2f'(x^2) + 2x\, f''(x^2)\frac{d}{dx}x^2$

+3 $= 2f'(x^2) + (2x)^2 f''(x^2)$

When $x = 1$, $h'(1) = 2f'(1) = 2(3) = 6$ +1

$h''(1) = 2f'(1) + 4f''(1)$

$= 2(3) + 4(5) = 26$ +1

$h'(1) =$ 6, $h''(1) =$ 26

8. Let $f(x) = \frac{x}{\sqrt{3 + x^2}}$.

7 pts

(a) Compute $f'(x)$ and simplify your answer by writing it as a single fraction.

(b) Find the linear approximation $L(x)$ to $f(x)$ at $x = 1$.

a) $f'(x) = \frac{\sqrt{3+x^2}\frac{d}{dx}x - x\frac{d}{dx}\sqrt{3+x^2}}{(\sqrt{3+x^2})^2}$

+3 $= \frac{\sqrt{3+x^2} - x\frac{1}{2\sqrt{3+x^2}}(2x)}{3+x^2}$ +1

$= \frac{(3+x^2) - x^2}{(3+x^2)^{3/2}} = \frac{3}{(3+x^2)^{3/2}}$

Can also use $\frac{d}{dx}\left[x(3+x^2)^{-\frac{1}{2}}\right]$

b)

$L(x) = f(1) + f'(1)(x-1)$

$= \frac{1}{\sqrt{4}} + \frac{3}{4^{3/2}}(x-1)$

$= \frac{1}{2} + \frac{3}{8}(x-1)$

$= \frac{3x+1}{8}$ +3

(a) $f'(x) =$ $\frac{3}{(3+x^2)^{3/2}}$, (b) $L(x) =$ $\frac{3x+1}{8}$

14 pts 9. (a) Define what it means to say that a number b is a critical number of a function f.

A number b is a critical number of a function f if b is in the domain of f and either $f'(b) = 0$ or $f'(b)$ does not exist (+2)

(+2)

(they may al[illegible] include endpts

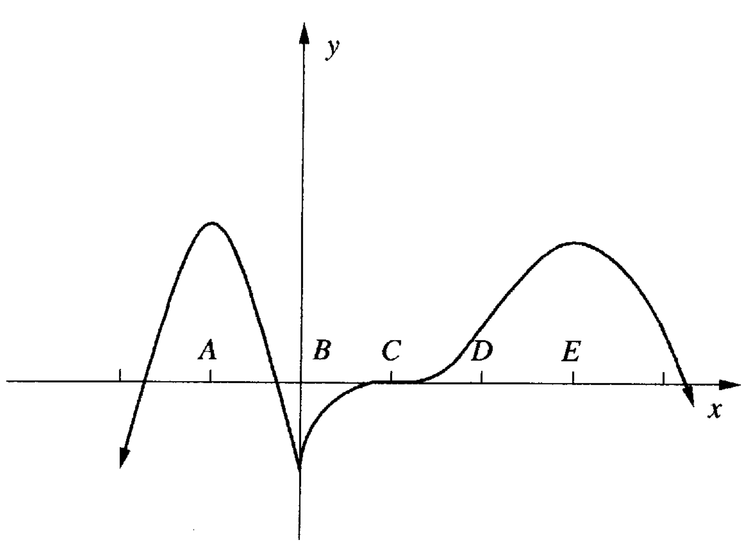
(b) If the graph of f is as pictured below, circle all the letters that label critical numbers of f on the x-axis. You do not need to provide an explanation.

(+5) A B C D E

(c) Circle all the letters which correspond to values on the x-axis where the function f has a local maximum. You do not need to provide an explanation.

(+5) A B C D E

one for each correct answer



Answer two of the following three questions. Indicate the question that is not to be graded by marking through this question on the front of the exam.

10. Suppose that f is a function that has derivatives of all orders at all values.

(a) Express $\frac{d}{dx}[xf(x)]$ in terms of f and its derivative.

(b) Prove using mathematical induction that

$$\frac{d^n}{dx^n}\ [xf(x)] = nf^{(n-1)}(x) + xf^{(n)}(x)$$

for all natural numbers $n = 1, 2, 3, \ldots$.

Recall that $f^{(0)} = f$ and for $n \geq 1$, $f^{(n)}$ is the nth derivative of f.

a) $\frac{d}{dx}[x f(x)] = f(x)\frac{d}{dx}x + x\frac{d}{dx}f(x)$

+3

$= f(x) + x f'(x)$

+1

b) The formula holds for $n=1$ by part (a) and $f(x) + x f'(x) = f^{(0)}(x) + x f^{(1)}(x)$. Suppose the formula holds for some natural number n. Then taking the derivative of each side and applying the product rule, we have

+3

+2 $\frac{d^{n+1}}{dx^{n+1}}[x f(x)] = \frac{d}{dx}\left[n f^{(n-1)}(x) + x f^{(n)}(x)\right]$

+3 $= n f^{(n)}(x) + f^{(n)}(x)\frac{d}{dx}x + x f^{(n+1)}(x)$

+1 $= (n+1) f^{(n)}(x) + x f^{(n+1)}(x)$. Thus the

+1 formula holds for $n+1$.

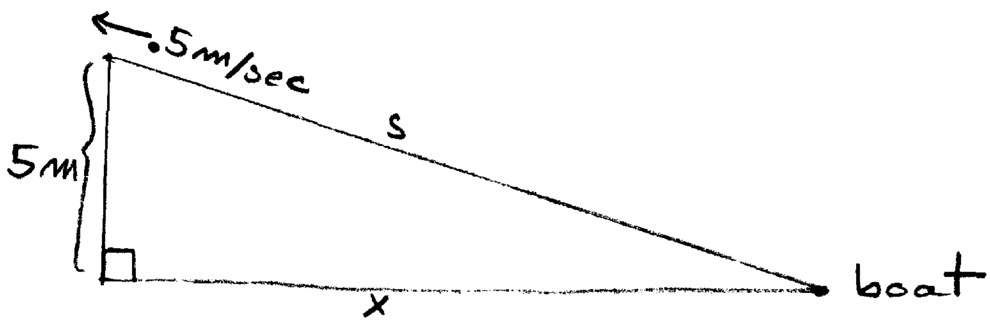
Therefore, by mathematical induction, the formula holds for all natural numbers n.

11. A boat is being pulled directly towards a dock by a rope. The rope is being pulled in at a rate of 0.5 meters/second and the height of the rope at the dock is 5 meters above the point where the rope is tied to the boat.

(a) Make a sketch which summarizes the information stated above and introduces variables for the problem in part (b).

(b) Use calculus to find the horizontal speed of the boat when the horizontal distance between the boat and dock is 12 meters.

a) +4



b) Given $\frac{ds}{dt} = -.5\, m/sec$, find $\frac{dx}{dt}$ when $x = 12\, m$.

+4 $s^2 = 5^2 + x^2$

$\frac{d}{dt} s^2 = \frac{d}{dt}(5^2 + x^2)$

$2s \frac{ds}{dt} = 2x \frac{dx}{dt}$

+3 $\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$

When $x = 12\, m$,

$s^2 = 5^2 + 12^2 = 13^2$

+2 $\therefore s = 13\, m$

$\frac{dx}{dt} = \frac{13}{12}(-.5)\, m/sec$

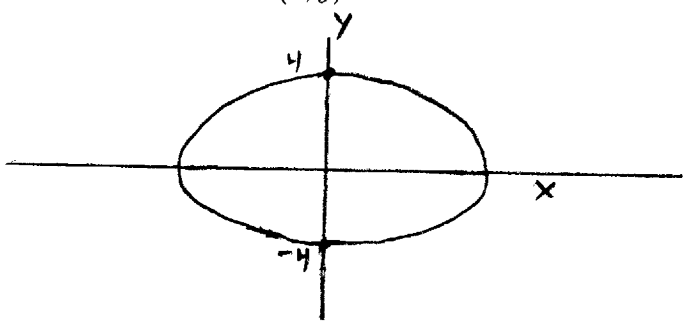
$= -\frac{13}{24}\, m/sec$ (This is the velocity)

Horizontal speed $= \frac{13}{24}\, m/sec.$ +1

14 pts 12. Consider the ellipse which is the set of all points (x, y) that satisfy the equation $4x^2 + 3y^2 = 48$.

(a) Find the derivative $\frac{dy}{dx}$ for all points on the ellipse where the derivative exists.

(b) Find all the points (x, y) on the ellipse and so that the slope of the tangent line to this curve at (x, y) is 2.



a) $\frac{d}{dx}(4x^2+3y^2) = \frac{d}{dx} 48$

(+4) $8x + 3(2y\frac{dy}{dx}) = 0$

$4x + 3y\frac{dy}{dx} = 0$

(+2) $\frac{dy}{dx} = -\frac{4x}{3y}$

b) $m = 2 \quad m = \frac{dy}{dx}$

$\therefore \; 2 = \frac{-4x}{3y}$

(+2) $y = -\frac{2x}{3}$

(+2) $4x^2 + 3\left(-\frac{2x}{3}\right)^2 = 48$

$x^2 + \frac{x^2}{3} = 12$

$\frac{4}{3}x^2 = 12$

$x^2 = 12\left(\frac{3}{4}\right) = 9$

(+2) $x = \pm 3$

If $x = 3$, $y = -\frac{2}{3}(3) = -2$

$x = -3$, $y = -\frac{2}{3}(-3) = 2$

Answer $(3,-2), (-3,2)$

(+2)