Multiple Choice Questions

5 pts each

Question					
1	A	B	C	**D**	E
2	A	B	C	D	**E**
3	A	**B**	C	D	E
4	A	B	**C**	D	E
5	A	**B**	C	D	E
6	**A**	B	C	D	E
7	A	**B**	C	D	E
8	A	B	C	D	**E**
9	A	**B**	C	D	E
10	A	B	C	**D**	E

(11) [10 points] Consider the function f defined by

$$f(x) = \begin{cases} x^2 - x & \text{if } x \leq 1 \\ \ln x & \text{if } x > 1. \end{cases}$$

Determine whether or not f is differentiable at $x = 1$ and explain your reasoning. If f is differentiable at $x = 1$, find $f'(1)$.

To be differentiable at $x = 1$, we need the limit of the difference quotient to exist, $$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}.$$	**Limits of difference quotient must exist: 3 pts**
Since the function is defined piecewise, it makes sense to consider the left and right limits separately, $$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1+} \frac{\ln(x)}{x - 1}$$ $$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x^2 - x}{x - 1}.$$	**Consider left and right limits separately: 2 pts**
We evaluate the limits to obtain $$\lim_{x\to 1+} \frac{\ln(x)}{x - 1} = 1, \qquad \lim_{x\to 1^-} \frac{x^2 - x}{x - 1} = 1.$$	**Correct left hand limit: 2 pts** **Correct right hand limit: 2 pts**
Students may evaluate the limit from the left by simplification. To evaluate the limit from the right, we must recognize that this is a difference quotient and use the derivative of $\ln(x)$.	
Since the two one-sided limits give the same value, the function is differentiable and the derivative is 1.	**Correct derivative at $x = 1$: 1 pt**

Many students evaluated the derivative of the functions $x^2 - x$ and $\ln(x)$ at $x = 1$ and observed that both gave the value 1. If we also observe that f is continuous at 1, these facts imply that f is differentiable at 1. Depending on how it was written, this argument is incomplete since it does not make clear how the derivatives of the functions $\ln(x)$ and $x^2 - x$ relate to the derivative of f. Such students received 2 points for continuity, 2 points for the derivative of each piece and 2 points for concluding differentiability for a maximum of 8 points.

(12) [10 points] Show that there is a number x such that $e^x = x^4$. Explain your reasoning carefully.

Let $f(x) = e^x - x^4$.	**Address the idea of creating a function from the equation: 2 pts**
$f(0) = e^0 - 0^4 = 1 > 0$	**Evaluating $f(x)$ at one endpoint giving a negative value: 2 pts**
$f(2) = e^2 - 2^4 \approx -8.611 < 0.$	**Evaluating $f(x)$ at one endpoint giving a positive value: 2 pts**
Since f is the sum of two continuous functions on $\mathbb{R}$, f is continuous.	**Statement that somehow f is continuous: 2 pts**
By the Intermediate Value Theorem, there must be a number $c \in (0, 2)$, so that $f(x) = 0$. This implies that $e^c = c^4$.	**Invocation of the IVT: 2 pts**

(13) [10 points] A particle is moving along a straight line. After t seconds of movement its position is $s(t) = te^{-\frac{t}{2}}$ meters.

(a) When is the particle at rest?

(b) When is the particle moving to the left?

(c) What is the total distance traveled by the particle over the time interval $[0, 4]$?

$s'(t) = e^{-t/2} - \dfrac{1}{2}te^{-t/2}$	**Correct derivative: 2 pts**
(a) The particle is at rest when its velocity is 0. Set $s'(t) = 0$, then $t = 2$.	**(a) Set $s'(t) = 0$: 1 pt** **Correct answer: 1 pt**
(b) The particle is moving to the left when $s'(t) < 0$, or when $t > 2$.	**(b) $s'(t) < 0$: 1 pt** **Correct answer: 1 pt**
(c) The particle moves right from $s(0)$ to $s(2)$ and then moves left from $s(2)$ to $s(4)$.	**(c) Process to compute total distance: 2 pts**
Total distance $= (s(2) - s(0)) + \lvert s(4) - s(2)\rvert$ $= (\left(\frac{2}{e} - 0\right) + \left(\frac{2}{e} - \frac{4}{e^2}\right)$ $= \frac{4e-4}{e^2} \approx 0.9302$	**Correct answer: 2 pts** Some students will compute net distance $(s(4) - s(0))$ in (c). In this case students may receive 2 points if they made the correct computation.

(14) [10 points] Consider the curve described by the equation

$$x^4e^y + 2\sqrt{y+1} = 3.$$

Find the equation of the tangent line to this curve at the point $(1, 0)$. Give the equation in the form $y = mx + b$.

$$\frac{d}{dx}\left(x^4e^y + 2(y+1)^{\frac{1}{2}}\right) = \frac{d}{dx}3$$
$$4x^3e^y + x^4e^y\frac{dy}{dx} + (y+1)^{-\frac{1}{2}}\frac{dy}{dx} = 0$$
$$\left(x^4e^y + (y+1)^{-\frac{1}{2}}\right)\frac{dy}{dx} = -4x^3e^y$$

Correct differentiation: 4 pts

$$\frac{dy}{dx} = -\frac{4x^3e^y}{x^4e^y + (y+1)^{-\frac{1}{2}}}$$

Correct $\frac{dy}{dx}$: 1 pt

$$\left.\frac{dy}{dx}\right|_{(1,0)} = -\frac{4}{2} = -2$$

Correct slope: 3 pts

$$y - 0 = -2(x-1)$$
$$y = -2x + 2$$

Correct equation: 2 pts

If the student does differentiation incorrectly, but solves correctly for $\frac{dy}{dx}$ and continues to use this form for the remainder of the problem, the student should lose the 4 differentiation points, but be eligible to receive all other points.

As long as the student computes the slope from their solution of the derivative, they should be eligible to receive points for the equation of the tangent line.

(15) [10 points] A runner is jogging due east at 8 kilometers per hour. A lighthouse is located 3 kilometers due north of her starting point. How fast is the distance between the runner and the lighthouse increasing when the runner is 5 kilometers away from the lighthouse? Include units in your answer and draw a picture of the situation, which includes the relevant quantities.

x = distance runner jogs in km y = distance from jogger to lighthouse in km	**Correct units: 2 pts**
Given: $\frac{dx}{dt} = 8$ Constraint: $x^2 + 3^2 = y^2$	**Correct constraint equation: 2 pt**
$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$	**Correct evaluation of $\frac{dy}{dt}$: 2 pts**
When $y = 5$, we have $x = 4$ and	**Correct computation of x: 1 pt**
$\frac{dy}{dt} = \frac{4}{5}(8) = \frac{32}{5}$ km/hr	**Correct answer: 2 pt**
Appropriate picture of a right triangle labelled correctly.	**Correct picture: 1 pt**