Exam 2  
Form A

Multiple Choice Questions

1. Suppose that \( f(x) = \begin{cases} 
  x + 2 & \text{if } x \leq 2 \\
  Ax + B & \text{if } 2 < x \leq 4 \\
  7x - 12 & \text{if } x > 4 
\end{cases} \)

Find the values of \( A \) and \( B \) which make \( f(x) \) continuous everywhere.

A. \( A = 6, \; B = 8 \)
B. \( A = -6, \; B = 8 \)
C. \( A = 6, \; B = -8 \)
D. \( A = -6, \; B = -8 \)
E. There is no solution.

**Solution:** We need \( \lim_{x \to a} f(x) = f(a) \) at both \( a = 2 \) and \( a = 4 \). \( \lim_{x \to 2} f(x) = 2 + 2 = 4 \) and \( \lim_{x \to 2^+} f(x) = 2A + B \) so we must have that \( 2A + B = 4 \).

Likewise at \( x = 4 \), \( \lim_{x \to 4^-} f(x) = 4A + B \) and \( \lim_{x \to 4^+} f(x) = 7 \cdot 4 - 12 = 16 \). This gives us two equations in two unknowns in which we can solve for \( A \) and \( B \).

\[
\begin{align*}
2A + B &= 4 \\
4A + B &= 16 \\
2A &= 12 \\
A &= 6 \\
2 \cdot 6 + B &= 4 \\
B &= -8
\end{align*}
\]

Thus, \( A = 6 \) and \( B = -8 \).
2. Find the horizontal asymptotes of \( f(x) = \frac{e^x}{1 + e^x} \).

A. \( y = 0 \)
B. \( y = 1 \)
C. \( y = 1/2 \)
D. \( y = 0 \) and \( y = 1 \)
E. \( y = -1 \)

**Solution:**

We need to find \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

\[
\lim_{x \to \infty} \frac{e^x}{1 + e^x} = \lim_{x \to \infty} \frac{e^x}{e^x + 1} = \lim_{x \to \infty} \frac{1}{e^{-x} + 1} = 1
\]

\[
\lim_{x \to -\infty} \frac{e^x}{1 + e^x} = \lim_{x \to -\infty} \frac{0}{1 + 0} = 0
\]

Thus, there are two horizontal asymptotes: \( y = 1 \) and \( y = 0 \).

3. Which of the following statements is true if \( f(x) \) is defined by

\[
f(x) = \begin{cases} 
\sin x & \text{if } x \leq -\pi/4 \\
\cos x & \text{if } x > -\pi/4 
\end{cases}
\]

A. \( f(x) \) is continuous at \( x = -\pi/4 \).

B. \( f(x) \) has a **jump discontinuity** at \( x = -\pi/4 \).

C. \( f(x) \) has an infinite discontinuity at \( x = -\pi/4 \).

D. \( f(x) \) is not defined at \( x = -\pi/4 \).

E. There is not enough information to determine the continuity of \( f(x) \) at \( x = -\pi/4 \).

**Solution:**

Checking \( \lim_{x \to -\pi/4} \sin x = -\frac{1}{\sqrt{2}} \) and \( \lim_{x \to -\pi/4} \cos x = \frac{1}{\sqrt{2}} \). Thus, both limits exist but are not equal. This means that \( f(x) \) has a jump discontinuity at \( x = -\pi/4 \).
4. Suppose that \( f \) is a continuous function on the interval \([0, 5]\) and we know that 
\[
\begin{align*}
  f(0) &= 1, &  f(1) &= -1, &  f(2) &= 1, &  f(3) &= -1, &  f(4) &= 1, &  f(5) &= -1.
\end{align*}
\]
Which of the following statements are true for any such \( f \)?

A. There are at least five solutions of the equation \( f(x) = 0 \) in the interval \([0, 5]\).
B. There are at most five solutions of the equation \( f(x) = 0 \) in the interval \([0, 5]\).
C. There are exactly five solutions of the equation \( f(x) = 0 \) in the interval \([0, 5]\).
D. The equation \( f(x) = -1 \) has at most three solutions in the interval \([0, 5]\).
E. The equation \( f(x) = 1 \) has exactly three solutions in the interval \([0, 5]\).

**Solution:** The function changes sign 5 times on the interval \([0, 5]\). Thus, by the Intermediate Value Theorem there must be at least five solutions to \( f(x) = 0 \) in that interval.

5. Find a formula for \( \frac{dy}{dx} \) in terms of \( x \) and \( y \), where \( x^2 + xy + y^2 = 1 \).

A. \( \frac{dy}{dx} = -\frac{2x + y}{x + y} \)
B. \( \frac{dy}{dx} = -\frac{2x}{x + 2y} \)
C. \( \frac{dy}{dx} = \frac{2x + y}{x + 2y} \)
D. \( \frac{dy}{dx} = -\frac{2x + y}{2y} \)
E. \( \frac{dy}{dx} = -\frac{2x + y}{x + 2y} \)

**Solution:**
\[
\begin{align*}
  x^2 + xy + y^2 &= 1 \\
  2x + \left( y + x \frac{dy}{dx} \right) + 2y \frac{dy}{dx} &= 0 \\
  (x + 2y) \frac{dy}{dx} &= -2x - y \\
  \frac{dy}{dx} &= -\frac{2x + y}{x + 2y}
\end{align*}
\]
6. Find all values of $x$ where $f'''(x) = 0$ when $f(x) = xe^{2x}$.

- A. $x = 3/2$
- B. $x = 2/3$
- C. $x = -3/2$
- D. $x = -2/3$
- E. $x = -1$

**Solution:**

$$f(x) = xe^{2x}$$
$$f'(x) = e^{2x} + 2xe^{2x}$$
$$f''(x) = 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x}$$
$$f'''(x) = 8e^{2x} + 4e^{2x} + 8xe^{2x} = (12 + 8x)e^{2x}$$

Setting $f'''(x) = 0$ gives $(12 + 8x)e^{2x} = 0$ which means $12 + 8x = 0$ giving us that $x = -3/2$.

7. Find $f'(x)$ in terms of $g'(x)$ where $f(x) = [g(x)]^4$.

- A. $f'(x) = 4[g(x)]^3$
- B. $f'(x) = 4[g(x)]^3g'(x)$
- C. $f'(x) = 4[g'(x)]^3$
- D. $f'(x) = 4[g(x)]^3[xg' + g]$
- E. $f'(x) = 4g'(x)$

**Solution:** From the Chain Rule,

$$f'(x) = 4[g(x)]^3g'(x).$$
8. Find the derivative of \( g(t) = \tan(\cos(2t)) \).
   A. \( g'(t) = 2 \sin(2t) \sec^2(\cos(2t)) \)
   B. \( g'(t) = \sin(2t) \sec^2(\cos(2t)) \)
   C. \( g'(t) = -\sin(2t) \sec^2(\cos(2t)) \)
   D. \( g'(t) = -2 \sin(2t) \sec^2(\cos(2t)) \)
   E. \( g'(t) = -2 \sec^2(\sin(2t)) \)

   **Solution:** This is again using the Chain Rule.
   \[ g'(t) = \sec^2(\cos(2t))(-\sin(2t))(2) = -2 \sin(2t) \sec^2(\cos(2t)). \]

9. Find the derivative of \( g(x) = x^5 \ln(9x) \).
   A. \( g'(x) = x^4(1 + 5 \ln(9x)) \)
   B. \( g'(x) = 1 + \frac{\ln(9x)}{9x} \)
   C. \( g'(x) = x^4 \left( \frac{1}{9} + 5 \ln(9x) \right) \)
   D. \( g'(x) = \frac{5}{9} x^3 \)
   E. \( g'(x) = x^4(5 \ln(9x) - 1) \)

   **Solution:** This is the Product Rule.
   \[ g'(x) = 5x^4 \ln(9x) + x^5 \frac{1}{9x} (9) = 5x^4 \ln(9x) + x^4 = x^4(1 + 5 \ln(9x)). \]
10. Differentiate

\[ f(x) = \frac{x^6}{1-x^5} \]

A. \[ f'(x) = \frac{6x^5}{1-5x^4} \]

B. \[ f'(x) = \frac{(x^5 - 1)^2}{x^3(6x - 5)} \]

C. \[ f'(x) = \frac{x^5(6x - 5)}{(1-x^5)^2} \]

D. \[ f'(x) = \frac{x^5(1-x^5)}{(6-x^5)^2} \]

E. \[ f'(x) = \frac{x^5(6-x^5)}{(1-x^5)^2} \]

**Solution:** This is the Quotient Rule.

\[ f'(x) = \frac{(1-x^5)6x^5-x^6(-5x^4)}{(1-x^5)^2} = \frac{6x^5 - 6x^{10} + 5x^{10}}{(1-x^5)^2} = \frac{x^5(6-x^5)}{(1-x^5)^2}. \]

11. Suppose that \( F(x) = f(g(x)) \) and \( g(14) = 2, g'(14) = 5, f'(14) = 15, \) and \( f'(2) = 11. \) Find \( F'(14) \)

A. \( F'(14) = 140 \)

B. \( F'(14) = 55 \)

C. \( F'(14) = 24 \)

D. \( F'(14) = 20 \)

E. \( F'(14) = 17 \)

**Solution:** From the Chain Rule, \( F'(x) = f'(g(x))g'(x) \), so

\[ F'(14) = f'(g(14))g'(14) = f'(2)g'(14) = 11 \times 5 = 55. \]
12. If \( f \) and \( g \) are continuous functions with \( f(9) = 6 \) and \( \lim_{x \to 9} [2f(x) - g(x)] = 9 \), find \( g(9) \).

A. \( g(9) = 21 \)
B. \( g(9) = 24 \)
C. \( g(9) = 3 \)
D. \( g(9) = 15 \)
E. \( g(9) = 12 \)

**Solution:** We are given that \( \lim_{x \to 9} [2f(x) - g(x)] = 9 \). Since \( f \) and \( g \) are continuous and \( f(9) = 6 \), we have

\[
2 f(9) - g(9) = 9
\]

\[
12 - g(9) = 9
\]

\[
g(9) = 3
\]

Free Response Questions

13. Find the derivatives of the following functions.

(a) \( f(x) = \ln(\cos(2x)) \).

**Solution:** By the Chain Rule

\[
f'(x) = \frac{\cos(2x)}{-\sin(2x)}(2) = \frac{2\sin(2x)}{-\cos(2x)} = -2\tan(2x).
\]

(b) \( g(x) = \frac{4}{x^5} - \frac{8}{x^4} - \frac{3}{x^3} + 700 \).

**Solution:** First, rewrite \( g(x) \).

\[
g(x) = 4x^{-5} - 8x^{-4} - 3x^{-3} + 700
\]

\[
g'(x) = -20x^{-6} + 32x^{-5} + 9x^{-4}
\]

\[
= -\frac{20}{x^6} + \frac{32}{x^5} + \frac{9}{x^4}
\]
(c) \( h(x) = 4 \ln(x^2 e^x) \).

**Solution:** First, rewrite \( h(x) \).

\[
\begin{align*}
h(x) &= 4 \left( \ln(x^2) + \ln(e^x) \right) \\
&= 8 \ln x + x \\
h'(x) &= \frac{8}{x} + 1
\end{align*}
\]

14. (a) Find the equation of the tangent line to \( 2xy^2 - 5x^2 y + 192 = 0 \) at the point \((4, 4)\).

**Solution:** We need to find \( \frac{dy}{dx} \bigg|_{(4,4)} \).

\[
\begin{align*}
2xy^2 - 5x^2 y + 192 &= 0 \\
2y^2 + 4xy \frac{dy}{dx} - 10xy - 5x^2 \frac{dy}{dx} &= 0 \\
(4xy - 5x^2) \frac{dy}{dx} &= 10xy - 2y^2 \\
\frac{dy}{dx} &= \frac{10xy - 2y^2}{4xy - 5x^2}
\end{align*}
\]

Now,

\[
\frac{dy}{dx} \bigg|_{(4,4)} = \frac{10(4)(4) - 2(4)^2}{4(4)(4) - 5(4)^2} = -8
\]

So the equation of the tangent line is

\[
y = 4 - 8(x - 4).
\]

(b) Find \( \lim_{x \to 0} \frac{\sin(3x)}{7x} \)

**Solution:**

\[
\lim_{x \to 0} \frac{\sin(3x)}{7x} = \frac{3}{7} \lim_{x \to 0} \frac{\sin(3x)}{3x} = \frac{3}{7} \times 1 = \frac{3}{7}.
\]
15. Let \( f(x) = \frac{x^2}{x+6} \).

(a) Find the derivative \( f'(x) \).

**Solution:** This is the Quotient Rule.

\[
f'(x) = \frac{(x+6)(2x) - x^2(1)}{(x+6)^2} = \frac{x^2 + 12x}{(x+6)^2}.
\]

(b) Find the equation of the tangent line to \( f(x) \) at the point where \( x = 4 \).

**Solution:** We need \( f(4) \) and \( f'(4) \). \( f(4) = \frac{16}{10} = \frac{8}{5} \). Meanwhile,

\[
f'(4) = \frac{16 + 48}{100} = \frac{16}{25}.
\]

The equation of the tangent line is then

\[
y - \frac{8}{5} = \frac{16}{25} (x - 4).
\]
16. This problem concerns the definition of the derivative using limits.

(a) State the formal definition of the derivative of a function $f(x)$ at the point $x = a$.

Hint: Your definition should involve a limit.

**Solution:**

The derivative of the function $f(x)$ at the point $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{s \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

(b) Using the formal definition of derivative and the limit laws, find the derivative of the function $f(x) = 2x^2 - 1$. An answer that is unsupported or uses differentiation rules will receive **no credit**.

**Solution:**

$$f'(a) = \lim_{h \to 0} \frac{2(a + h)^2 - 1 - (2a^2 - 1)}{h}$$

$$= \lim_{h \to 0} \frac{2a^2 + 4ah + 2h^2 - 1 - 2a^2 + 1}{h}$$

$$= \lim_{h \to 0} \frac{4ah + 2h^2}{h}$$

$$= \lim_{h \to 0} 4a + 2h$$

$$f'(a) = 4a$$