Name: ______________________________ Section and/or TA: ______________

Do not remove this answer page. You will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or communication capabilities is permitted. You may not use a phone or other communication device during the exam.

The exam consists of 12 multiple choice questions that are worth 5 points each and 4 free response questions that are worth 10 points each. You should work the multiple choice questions on the question page. After you have checked your work carefully, record your answers by completely filling in the circle below that corresponds to your answer. If you must change your answer, make a note on the front of the exam. Be sure to check carefully when you transfer your answers to the cover sheet.

Show all work to receive full credit on the free response problems. You do not need to compute a decimal approximation to your answer. For example, the answer 4π is preferred to 12.57.

Multiple Choice Questions

1	A	B	C	D	E	**7**	A	B	C	D	E
2	A	B	C	D	E	**8**	A	B	C	D	E
3	A	B	C	D	E	**9**	A	B	C	D	E
4	A	B	C	D	E	**10**	A	B	C	D	E
5	A	B	C	D	E	**11**	A	B	C	D	E
6	A	B	C	D	E	**12**	A	B	C	D	E

SCORE

Multiple Choice	13	14	15	16	Total Score
60	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Let $f(x) = x\cos(2x)$. Find $f'(\pi/2)$.

   A. -1 B. π C. $-\pi$ D. 1 E. $1-\pi$

Solution: Use the product and chain rules to find that

$$f'(x) = 1\cos(2x) + x(-\sin(2x)) \cdot 2 = \cos(2x) - 2x\sin(2x).$$

Substituting $x = \pi/2$, $f'(\pi/2) = \cos(\pi) - \pi\sin(\pi) = -1$.

Compare WS 2.8, #1a)

2. (5 points) If $f(x) = xe^{2x}$, find $f''(x)$.

   A. $(4x+3)e^{2x}$

   B. $(4x+4)e^{2x}$

   C. $(4x+2)e^{2x}$

   D. $4xe^{2x}$

   E. $(4x+1)e^{2x}$

Solution: We use the product and chain rule to find $f'(x) = (2x+1)e^{2x}$ and then $f''(x) = 2(2x+1)e^{2x} + 2e^{2x} = (4x+4)e^{2x}$.

Compare WW2.14 #2.

3. (5 points) Suppose f is differentiable at 2 with $f(2) = 2$ and $f'(2) = 3$. Let $g(x) = \dfrac{f(x)}{x^2}$ and find $g'(2)$.

A. 1/8 B. 1/2 C. 3/8 D. 0 *E.* 1/4

Solution: From the quotient rule, $g'(x) = \dfrac{x^2 f'(x) - 2x f(x)}{x^4}$. Setting $x = 2$, $g'(2) = \dfrac{4 \cdot 3 - 2 \cdot 2 \cdot 2}{16} = 1/4$.

Compare WW2.4,2.6#6

4. (5 points) Find the tangent line to $f(x) = x \ln(x)$ at $x = 1$.

   A. $y = x - 1$

   B. $y = x + 1$

   C. $y = x$

   D. $y = -x + 1$

   E. $y = -x - 1$

Solution: The derivative of $f(x) = x \ln(x)$ is $f'(x) = \ln(x) + x\frac{1}{x} = \ln(x) + 1$. Setting $x = 1$, the slope of the tangent line at 1 is $f'(1) = 1$ and the tangent line passes through $(1, 0)$. The equation is $y = x - 1$.

Compare Worksheet §2.10 #3b)

5. (5 points) Assume that x and y are related by the equation $y^2 + ax = 3a^2$ where $a \neq 0$ is a nonzero constant. Find dy/dx at the point $(2a, a)$.

A. $1/2$ *B.* $-1/2$ C. $-a/2$ D. $a/2$ E. 1

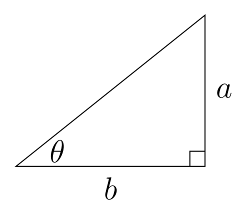
Solution: Use implicit differentiation. Assuming that $y = y(x)$ is a function of x, we find the derivative by differentiating the equation, $y^2 + ax = 3a^2$ to find

$$2y\frac{dy}{dx} + a = 0.$$

Solving for dy/dx, we have $\frac{dy}{dx} = -a/(2y)$ and subsituting $(x, y) = (2a, a)$, $dy/dx = -a/(2a) = -1/2$.

Compare WS 1.4, #3, WW1.4#2,3

6. (5 points) In the right triangle at right, express b in terms of a and the angle θ.



A. $b = a\cos(\theta)$

B. $b = a\tan(\theta)$

C. $b = a/\tan(\theta)$

D. $b = a\sec(\theta)$

E. $b = a\sin(\theta)$

Solution: We have $\tan(\theta) = a/b$ and solving for b gives $b = a/\tan(\theta)$.

Compare WS §2.8, #4

7. (5 points) Suppose $\cos(\theta) = t$ and $0 \leq \theta \leq \pi$. Express $\sin(\theta)$ without using trigonometric or inverse trigonometric functions.

   A. $\sin(\theta) = -\sqrt{1-t^2}$

   B. $\sin(\theta) = \sqrt{1-t^2}$

   C. $\sin(\theta) = -\sqrt{1+t^2}$

   D. $\sin(\theta) = -t/\sqrt{t^2-1}$

   E. $\sin(\theta) = \sqrt{1+t^2}$

Solution: We use the Pythagorean identity to write $\sin(\theta) = \pm\sqrt{1-\sin^2(\theta)}$. We choose the positive square root since $\sin(\theta) \geq 0$ for $0 \leq \theta \leq \pi$. Finally substituting t for $\cos(\theta)$ gives $\sin(\theta) = \sqrt{1-t^2}$.

Compare WS2.8 #3, WW2.8#2

8. (5 points) Let $f(x) = \arctan(3x)$. Find $f'(x)$.

   A. $\dfrac{1}{1+x^2}$

   B. $\dfrac{3}{1+x^2}$

   C. $\dfrac{1}{1+9x^2}$

   D. $\dfrac{3}{1+9x^2}$

   E. $\dfrac{1}{1+3x^2}$

Solution: We recall the derivative of arctan, $\arctan'(x) = \frac{1}{1+x^2}$. We use the chain rule to write

$$f'(x) = \frac{1}{1+(3x)^2}(3x)' = \frac{3}{1+9x^2}.$$

In the last step, we used that $(3x)^2 = 3^2x^2 = 9x^2$.

Compare WW2.12#12

9. (5 points) Assume the acceleration of gravity is 10 meters/second^2 in the downward direction. A ball is thrown upward from a position 100 meters above the ground and with a speed of 30 meters/second. Find the height of the ball above the ground 3 seconds after it is thrown.

   A. 235 meters

   B. 100 meters

   C. 145 meters

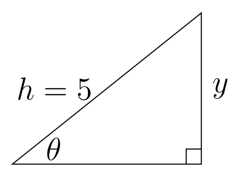
   D. 190 meters

   E. 55 meters

Solution: The height at time t is $h(t) = 100 + 30t - 5t^2$. Setting $t = 3$, we have $h(3) = 100 + 90 - 45 = 145m$.

Compare WS§3.1#4.

10. (5 points) In the triangle pictured at right, $h = 5$ is fixed and y is varying. Find the derivative $d\theta/dy$ when $y = 4$.



A. 1/4 B. 1/5 C. 3/4 D. 4/3 *E. 1/3*

Solution: We may write $\theta = \arcsin(y/5)$. We use that the derivative of arcsin is $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$ and the chain rule to find

$$\frac{d\theta}{dy} = \frac{1}{5}\frac{1}{\sqrt{1-(y/5)^2}}.$$

Substituting $y = 4$ gives

$$\frac{d\theta}{dy} = \frac{1}{5} = \frac{1}{\sqrt{1-16/25}} = \frac{1}{5}\frac{1}{\sqrt{9/25}} = \frac{1}{3}.$$

Compare WS§2.12#8

11. (5 points) Let $f(x) = \cos(x)$ and find $f^{(101)}(x)$, the 101st derivative of f.

A. $\sin(x)$

B. 0

C. $-\sin(x)$

D. $-\cos(x)$

E. $\cos(x)$

Solution:
Compare WW2.14 #6.

12. (5 points) Find $Q(x)$, the quadratic approximation to $f(x) = e^{2x}$ at $x = 0$.

A. $Q(x) = 1 + 2x + 4x^2$

B. $Q(x) = 1 + x + x^2$

C. $Q(x) = 1 + 2x + x^2$

D. $Q(x) = 1 + 2x + 2x^2$

E. $Q(x) = 1 + x + 2x^2$

Solution: If $f(x) = e^{2x}$, then $f'(0) = 2$ and $f''(0) = 4$. Then $Q(x) = f(0) + f'(0)x + \frac{1}{2}f''(x)x^2 = 1 + 2x + 2x^2$.
Compare WW§3.4.1-3#4

Free response questions, show all work

13. (10 points) (a) Find the equation of the tangent line to $y = x^2$ at the point (a, a^2).

(b) Use your answer to part a) to find an equation that a must satisfy if the tangent line to $y = x^2$ at (a, a^2) passes through the point $(2, 0)$.

(c) Solve the equation from part b) and find all tangent lines to the graph of $y = x^2$ which pass through the point $(2, 0)$.

Solution: a) The line tangent to $y = x^2$ at (a, a^2) has slope $2a$ and passes through (a, a^2). The slope-point form of the line is $y - a^2 = 2a(x - a)$. Writing this in slope-intercept form gives $y = 2ax - a^2$.

b) If a tangent line passes through $(2, 0)$, then substituting $(x, y) = (2, 0)$ into the equation of the tangent line, we will have that a must satisfy

$$0 = 4a - a^2.$$

c) Solving $4a - a^2 = a(4 - a) = 0$, we have $a = 0, 4$. We may substitute these values of a into our answer for part a) to find the tangent lines. The tangent line for $a = 0$ is $y = 0$. The tangent line for $a = 4$ is $y = 8x - 16$.

Compare WA3

Grading: a) Correct slope (1 point), (a, a^2) lies on line (1 point), equation (1 point)

b) Require line to pass through (2,0) (1 point), equation $4a - a^2 = 0$ (1 point)

c) Values for a (1 point, each), tangent lines (2 points 1st line, 1 point 2nd line)

14. (10 points) (a) Find all values b so that the point $(2, b)$ lies on the ellipse $x^2 + 3y^2 = 7$.

(b) Suppose that $x^2 + 3y^2 = 7$. Find $\dfrac{dy}{dx}$ in terms of x and y.

(c) For each of the points $(2, b)$ on the ellipse that you found in part a), find the equation of the line that is tangent to the ellipse at $(2, b)$. Simplify to write the equation of each tangent line in slope-intercept form.

Solution: a) We substitute $(2, b)$ into $x^2 + 3y^2 = 7$ and solve to find $4 + 3b^2 = 7$ and then $b^2 = 1$. Thus $b = \pm 1$ are the values for which $(2, b)$ lies on the ellipse.

b) Differentiating implicitly with respect to x gives $2x + 6y\frac{dy}{dx} = 0$. Solving for the derivative, we find

$$\frac{dy}{dx} = -\frac{x}{3y}.$$

c) Finding the tangent lines. At $(2, 1)$, we have that $\frac{dy}{dx} = -\frac{2}{3 \cdot 1} = -2/3$. The tangent line is the line through $(2, 1)$ with slope $-2/3$. The equation of this line is

$$y - 1 = -\frac{2}{3}(x - 2) \quad \text{or} \quad y = -\frac{2}{3}x + \frac{7}{3}.$$

At $(2, -1)$, the slope is $\frac{dy}{dx} = -\frac{2}{3(-1)} = 2/3$. The tangent line is the line through $(2, -1)$ and with slope $2/3$. The equation is

$$y + 1 = \frac{2}{3}(x - 2) \quad \text{or} \quad y = \frac{2}{3}x - \frac{7}{3}.$$

Compare WS1.6 #7,8, WSR2#3.

Grading: a) Equation for b (1 point), values (1 point each).

b) Differentiate implicitly (2 points),

c) Find slopes (1 point for each slope), tangent line (1 point each + 1 point for simplifying both correctly)

15. (10 points) (a) State the mean value theorem.

(b) Let $f(x) = \sin(2x)$. Find the largest possible value of $f'(x)$ for x in $(-\infty, \infty)$.

(c) Use the mean value theorem on the interval $[0, 1/6]$ and your answer to part b) to find a value M so that $f(1/6) \leq M$.

Solution: a) Theorem: If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a number c in the interval (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

b) Since $f'(x) = 2\cos(2x)$ and $-1 \leq \cos(2x) \leq 1$, we have $f'(x) \leq 2$.

c) Applying the mean value theorem to f on the interval $[0, 1/6]$ and using our upper bound for f' in part b) we have

$$\frac{f(1/6) - f(0)}{1/6 - 0} = f'(c) \leq 2.$$

Solving this inequality and using that $f(0) = 0$, we have

$$f(1/6) \leq 2/6 = 1/3.$$

We may use $M = 1/3$.

Compare WSR2 #5

a) Continuous on closed interval (1 point), differentiable on open interval (1 point), there exist c with $(f(b) - f(a))/(b - a) = f'(c)$ (1 point), c lies in $[a, b]$ (1 point).

b) $\cos(x) \leq 1$ for all x (1 point), $f'(x) = 2\cos(2x)$, so $f'(x) \leq 2 = M$ (1 point).

c) The function $f(x)$ is differentiable everyhwere (1 point), $(f(1/6)-f(0))/(1/6-0) = f'(c)$ (1 point), $f(0) = \sin(0) = 0$ (1 point), so $f(1/6) \leq \frac{1}{6}M = 1/3$ (1 point).

16. (10 points) (a) Find the linear and quadratic approximations to $f(x) = \sqrt{x}$ at 25.

   (b) Use the linear approximation to find an approximate value for $\sqrt{22}$. Your answer should be correctly rounded to three decimal places or more accurate.

   (c) Use the quadratic approximation to find an approximate value for $\sqrt{29}$. Your answer should be correctly rounded to three decimal places or more accurate.

   Hint: In parts b) and c) you may check if your answer is reasonable by comparing it with the values of $\sqrt{22}$ and $\sqrt{29}$ from a calculator. However, you will need to show that you found your approximations using the linear and quadratic approximations to receive full credit for this problem.

Solution: We compute $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$. Thus $f(25) = 5$, $f'(25) = 1/10$, $f''(25) = -1/500$.

Then $L(x) = 5 + (x-25)/10$ and $Q(x) = 5 + (x-25)/10 - (x-25)^2/1000$.

Substituting 22 into L gives $\sqrt{22} \approx L(22) = 5 - 3/10 = 4.7$. Substituting 29 into Q gives $\sqrt{29} \approx Q(29) = 5 + 4/10 - 16/1000 = 5.384$.

Compare WS3.1-3#2, WSR2 #7

Grading: a) Value for $f(25)$ (1 point), $f'(x)$ (1 point), $f''(x)$ (1 point) linear approximation $\ell(x)$ (2 points), quadratic approximation (2 points)

b) value for linear approximation at 22 (1 point)

c) Value for quadratic approximation at 29 (1 point), (1 additional point if answers for b) and c) are accurate to at least three decimal places.