Calculus I
Exam 2

Common exam
7 March 2006

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*.
Each question is followed by space to write your answer. Please lay out your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name Key
Section ________
Last four digits of student identification number ________

Question	Score	Total
p. 1		14
p. 2		14
p. 3		14
p. 4		14
p. 5		14
Q11		14
Q12		14
Q13		14
Free	2	2
		100

1. Suppose that $\cos(x) = 12/13$ and x lies in the interval $[-\pi, 0]$, find $\sin(x)$.

(2) $\sin^2 x + \cos^2 x = 1$

$$\sin x = \pm\sqrt{1-\cos^2 x}$$

If $\cos x = 12/13$,

$$\sin x = \pm\sqrt{1 - 144/169}$$

(3) $$= \pm 5/13$$

(2) Choose negative sign since x in $[-\pi, 0]$

$\sin(x) =$ $-5/13$

2. Use the limit laws to compute the following limits:

(a) $\lim_{x\to 0}\left(\frac{\cos(x)-1}{x}\right)^2 =$ 0, (b) $\lim_{x\to 1}\left(\frac{\sin(x-1)}{x^2-1}\right) =$ $1/2$

$$\lim_{x\to 0}\left(\frac{\cos(x)-1}{x}\right)^2 = \left(\lim_{x\to 0}\frac{\cos x - 1}{x}\right)^2$$

$$= 0 \quad (3)$$

$$\lim_{x\to 1}\left(\frac{\sin(x-1)}{x^2-1}\right) = \lim_{x\to 1}\frac{\sin(x-1)}{x-1}\cdot\lim_{x\to 1}\frac{1}{x+1} \quad (2)$$

$$= 1 \times 1/2 = 1/2$$

(1) (1)

a - Give full credit for answer.

3. Let $g(t) = 2t + \sin(2t)$. Find all values of t in the interval $[0, \pi]$ so that the tangent line to the graph of g at the point $(t, g(t))$ is parallel to the line $y = t + 4$.

$g'(t) = 2 + 2\cos(2t)$ 3

Want $g'(t) = 1$,

$2 + 2\cos 2t = 1$

$2\cos 2t = -1$

$\cos 2t = -1/2$ 2

$2t = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$

$t = \pi/3, 2\pi/3$ 1 each.

$\pi/3,\ 2\pi/3$

4. Compute the following derivatives:

(a) If $f(x) = \sqrt{9 + x^2}$, find $f'(4)$. (b) If $g(x) = \sin(\frac{\pi\sqrt{x+3}}{2})$, find $g'(1)$.

$f'(x) = 2x \frac{1}{2} (9+x^2)^{-1/2}$ 2

$= \frac{x}{\sqrt{9+x^2}}$

$f'(4) = 4/5$ 1

$g'(x) = \cos(\frac{\pi}{2}\sqrt{x+3})$

$\times \frac{\pi}{4} (x+3)^{-1/2}$ 2

$g'(1) = \frac{\pi}{4} 4^{-1/2} \cos(\pi)$

$= -\frac{\pi}{8}$ 2.

(a) $f'(4) =$ $4/5$, (b) $g'(1) =$ $-\pi/8$.

$-1/2$

5. Consider the hyperbola with equation $\frac{y^2}{25} - \frac{x^2}{16} = \cancel{1}$. Determine the equation of the line which is tangent to this hyperbola at the point $(3, \frac{5}{4})$. Put your answer in the form $y = mx + b$.

Differentiate implicitly,

(2) $\frac{2y}{25}\frac{dy}{dx} - \frac{x}{8} = 0$

(1) $\frac{dy}{dx} = \frac{x}{8} \cdot \frac{25}{2y} = \frac{25}{16}\frac{x}{y}$

If $(x, y) = (3, 5/4)$

$\frac{dy}{dx} = \frac{25}{16} \cdot \frac{12}{5} = \frac{5 \cdot 3}{4} = \frac{15}{4}$ (1)

$y - \frac{5}{4} = \frac{15}{4}(x - 3)$. (2)

$y = \frac{15}{4}x - \frac{\cancel{45}}{4} + \frac{5}{4}$

$= \frac{15}{4}x - \frac{40}{4}$

$= \frac{15}{4}x - 10$

$y =$ $\frac{15}{4}x - 10$ (1)

6. A particle is moving on a straight line so that after t hours, the particle is $s(t) = 30t - t^3 + 50$ kilometers west of Rupp Arena. Find the velocity and the acceleration of the particle after 30 minutes.

30 minutes is $t = 1/2$ hour. (1)

$s'(t) = 30 - 3t^2$. (2)

$s''(1/2) = -3$

Velocity at $t = 1/2$,

$s'(1/2) = 30 - 3/4 = 29\,1/4$

Acceleration

$s''(t) = -6t$ (2)

(1) (1)

$29\,1/4$ km/h -3 km/hr^2

7. Find the second derivative of $f(x) = \sin(1 - x^3)$.

$$f'(x) = -3x^2 \cdot \cos(1-x^3) \quad 4$$

$$f''(x) = -6x \cos(1-x^3) - 3x^2(-3x^2)(-\sin(1-x^3)).$$

$$= -6x \cos(1-x^3) + 9x^4 \sin(1-x^3). \quad 3$$

$f''(x) =$ $-6x \cos(1-x^3) - 9x^4 \sin(1-x^3)$.

8. Let $f(x) = \sqrt[3]{x+1}$.

(a) Find the linear approximation (or linearization) L of f at 0. (b) Use L to approximate $\sqrt[3]{1.3}$.

Linear approximation is $L(x) = f(0) + f'(0)(x - 0)$

$$f'(x) = \frac{d}{dx}(x+1)^{1/3} = \frac{1}{3}(x+1)^{-2/3}. \quad 2$$

1 $\quad f'(0) = \frac{1}{3} \quad$ 1 $\quad f(0) = 1.$

$$L(x) = 1 + \frac{1}{3}x \quad 2$$

$$L(1.3) = 1 + \frac{1}{3}0.3 \quad 2$$

(a) $L(x) =$ $1 + \frac{1}{3}x$ (b) $\sqrt[3]{1.3} =$ 1.1

9. Find the absolute maximum value and the absolute minimum value of $f(x) = x^3 - 6x^2$ on the interval $[-1, 7]$.

$f'(x) = 3x^2 - 12x$ (2)

$= 3x(x-4).$

critical numbers at $x=0$ or $x=4$

And at endpoints -1, 7. (2)

compute values of f

$f(0) = 0$, $f(7) = 7^3 - 6 \cdot 7^2 = 7^2(7-6) = 49$

$f(-1) = -7$ $f(4) = 64 - 6 \cdot 16 = -32.$

(1) Test values

Maximum value 49 (1), Minimum value -32 (1)

10. Let f be a function that is differentiable, satisfies $8 \geq f'(x) \geq 5$ for all real numbers x, and $f(1) = 7$. Use the mean value theorem to find a number A so that $f(3) \geq A$.

By the mean value theorem,

$$\frac{f(3) - f(1)}{3-1} = f'(c) \quad (2) \quad \text{for some } c.$$

As $f(1) = 7$ and $8 \geq f'(c) \geq 5$, we have

$$\frac{f(3) - 7}{2} \geq 5. \quad (2)$$

$$f(3) \geq 10 + 7 = 17.$$

17 (3)

Answer two of the following three questions. Indicate the question that is not to be graded by marking through this question on the front of the exam.

11. (a) State Rolle's theorem. Use complete sentences.
    (b) State the mean value theorem. Use complete sentences.
    (c) Show how to use Rolle's theorem to prove the mean value theorem.

4

(a) If f is continuous on an interval $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there is a c in (a,b) so that $f'(c) = 0$.

(b) If f is continuous on an interval $[a,b]$, differentiable on a open interval (a,b), then there is a c in (a,b) so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

4

(c) If f be continuous on a closed interval $[a,b]$, differentiable on (a,b). and set

$$g(x) = f(x) - \frac{(f(b)-f(a))}{b-a}(x-a).$$

6

The g is cont. on $[a,b]$ and differentiable on (a,b).

$$g(a) = g(b) = f(a).$$

Thus by Rolle's theorem $g'(c) = 0$ for some c in (a,b). But $g'(c) = f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right)$.

So $f'(c) = \frac{f(b)-f(a)}{\ }$ ∎

12. Let C be a circle of radius 5 centimeters with center at the origin. A very speedy spider begins at the point $(5,0)$ and crawls around the circle C in the counterclockwise direction so that it completes 4 trips around the circle per minute.

(a) Find functions $(x(t), y(t))$ which give the location of the spider at time t minutes after it leaves the point (5,0).

(b) Give the location of the spider 20 seconds after it leaves the point $(5,0)$.

(c) Let $f(t)$ be the distance between the point $(1,0)$ and the spider at time t. Find $f(t)$.

(d) Find $f'(t)$.

a) In one minute, the spider completes 4 revolutions or 8π radians. Thus in t minutes, the spider completes $8\pi t$ (2) radians. Position at time t is

$$x(t) = 5\cos(8\pi t) \quad (2)$$

$$y(t) = 5\sin(8\pi t). \quad (2)$$

(b) 20 sec is $t = 1/3$ min.

$$x(1/3) = 5\cos(8\pi/3) = 5\cos(2\pi/3) = -5/2$$

$$y(1/3) = 5\sin(8\pi/3) = 5\sin(2\pi/3) = 5\sqrt{3}/2 \quad (2)$$

(c) $$f(t) = \sqrt{(5\cos(8\pi t)-1)^2 + (5\sin(8\pi t))^2} \quad (3)$$

$$= \sqrt{25\cos^2 8\pi t - 10\cos(8\pi t) + 1 + 25\sin^2(8\pi t)}$$

$$= \sqrt{26 - 10\cos(8\pi t)}$$

(Simplification not required)

(d) $$f'(t) = \frac{1}{2}(26 - 10\cos(8\pi t))^{-1/2} \cdot 80\pi \sin(8\pi t)$$

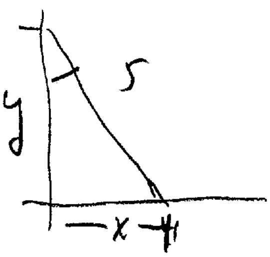
$$= \frac{40\pi \sin(8\pi t)}{\sqrt{26-10\cos(8\pi t)}} \quad (3)$$

13. A ladder of length 5 meters is leaning against a wall. The wall and floor meet in a right angle.

(a) If the base of the ladder is 4 meters from the wall, how high is the top of the ladder above the floor?

(b) Suppose the base of the ladder is moving away from the floor at the speed of 0.2 meters/second. Let θ be the angle between the ladder and the wall. Find the rate of change of θ when the base of the ladder is 4 meters away from the wall.

(c) In part (b), is the angle increasing or decreasing?



(a) $x^2+y^2=5^2$. if $x=4$, $y=\sqrt{5^2-4^2}=3$. meters 3

3 meters up the wall

b) $\sin\theta = \frac{x}{5}$ 1, $\cos\theta = y/5$. $\frac{dx}{dt} = 0.2$ m/sec. 1

$$\frac{d\theta}{dt}\cos\theta = \frac{1}{5}\frac{dx}{dt}.$$

$$\frac{d\theta}{dt} = \frac{1}{\cos\theta}\frac{1}{5}\frac{dx}{dt} \quad 3 \qquad \cos\theta = 4/5 \quad 1$$

$$= \frac{5}{3}\cdot\frac{1}{5}\cdot(0.2)\ \text{1/sec}$$

$$= \frac{2}{10}\,\frac{1}{3} = \frac{1}{15}\ 2\ \text{radians/sec}\ 1\ \text{units}$$

(c) ~~the~~ angle θ is increasing 2