(1) Given that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, use the limit laws to find the following limits.

(a) $\lim_{t \to 0} \frac{\sin(7t^2)}{3t}$.

(b) $\lim_{t \to 0^+} t\cot(3t)$.

4 (a) $\lim_{t \to 0} \frac{\sin(7t^2)}{3t} = \lim_{t \to 0} \left(\frac{7t^2}{3t}\right)\left(\frac{\sin(7t^2)}{7t^2}\right)$

$= \lim_{t \to 0} \frac{7}{3}t \cdot \lim_{t \to 0} \left(\frac{\sin(7t^2)}{7t^2}\right)$ (by the limit laws)

$= 0 \cdot 1 = 0$

4 (b) $\lim_{t \to 0} t \cot(3t) = \lim_{t \to 0} \left(\frac{3t}{\sin(3t)}\right)\left(\frac{\cos(3t)}{3}\right)$

$= \lim_{t \to 0} \left(\frac{3t}{\sin(3t)}\right) \lim_{t \to 0} \left(\frac{\cos(3t)}{3}\right) = \frac{1}{\lim_{t \to 0}\left(\frac{\sin(3t)}{3t}\right)} \cdot \frac{1}{3}$

(by the limit laws and continuity of $\frac{1}{3}\cos(3t)$ at $t=0$) $= \frac{1}{3}$

(a) 0 (b) $\frac{1}{3}$

(2) Find the following derivatives.

(a) $g'(t)$ at $t = 0$ when $g(t) = \dfrac{2t+3}{t^2+t+4}$.

(b) $h'(x)$ when $h(x) = 3\sin(4x^2) + 2\tan(x)$.

(c) $f'(x)$ when $f(x) = \left(\sqrt{2+x+x^4}\right)^3$.

4 (a) $g'(t) = \dfrac{(t^2+t+4)(2) - (2t+3)(2t+1)}{(t^2+t+4)^2}$.

Hence, $g'(0) = \dfrac{8-3}{4^2} = \dfrac{5}{16}$.

4 (b) $h'(x) = 24x\cos(4x^2) + 2\sec^2(x)$.
(by the chain rule)

4 (c) $f'(x) = \frac{3}{2}(1+4x^3)(2+x+x^4)^{\frac{1}{2}}$.
(by the chain rule)

(a) $g'(0) =$ $5/16$

(b) $h'(x) =$ $24x\cos(4x^2) + 2\sec^2(x)$

(c) $f'(x) =$ $\frac{3}{2}(1+4x^3)(2+x+x^4)^{1/2}$

(3) Find the following second derivatives:

(a) $g''(x)$ when $g(x) = (1-2x)^{-3}$.

(b) $h''(1)$ when $h(x) = (x-2)\sin(\pi x)$.

5 (a) $g'(x) = (-3)(-2)(1-2x)^{-4} = 6(1-2x)^{-4}$

$g''(x) = (-4)(-2)\,6\,(1-2x)^{-5} = 48(1-2x)^{-5}$

5 (b) $h'(x) = \sin(\pi x) + \pi(x-2)\cos(\pi x)$

4 $h''(x) = \pi\cos\pi x + \pi\cos(\pi x) - \pi^2(x-2)\sin\pi x.$

1 Hence $h''(1) = 2\pi\cos\pi + \pi^2(\pi-2)\sin\pi = -2\pi$

(since $\sin\pi = 0$, $\cos\pi = -1$)

(a) $g''(x) =$ $48(1-2x)^{-5}$

(b) $h''(1) =$ -2π

(4) A particle is traveling along a straight line. Its position after t seconds is given by $s(t) = 2t^3 - 21t^2 + 72t + 10$ meters.

(a) Find the velocity and acceleration of the particle at time t.

(b) Determine all times, t, where the velocity is zero.

7 (a) $v(t) = \frac{ds}{dt} = 6t^2 - 42t + 72.$

$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 12t - 42,$

3 (b) $v(t) = 6(t^2 - 7t + 12).$

$= 6(t-4)(t-3) = 0$ when

$t - 3 = 0, t - 4 = 0,$

$t = 3, 4$ seconds

(a) (i) $v(t) =$ $6t^2 - 42t + 72$ m/s (ii) $a(t) =$ $12t - 42$ m/s^2

(b) $t = 3, 4$ seconds

(5) Consider the curve given by the equation $y^3 + yx^4 = 2$. Find the equation of the tangent line to this curve at (1,1). Write your answer in the form $y = mx + b$.

7 $\frac{d}{dx}(y^3 + yx^4) = \frac{d}{dx}(2) = 0$

$\parallel$

$3y^2 \frac{dy}{dx} + \frac{dy}{dx} x^4 + 4x^3 y = 0$ (gather terms)

$(3y^2 + x^4)\frac{dy}{dx} = -4x^3 y$

Hence $\left.\frac{dy}{dx}\right)_{x=1} = \left.\frac{-4x^3y}{3y^2+x^4}\right)_{(1,1)} = -1$

3 point slope equation is $y - 1 = -(x-1)$

so $y = -x + 2$

Equation of tangent line: $y = -x + 2$

(6) The cost of making x campaign buttons for the upcoming presidential election is given by $C(x) = 2\sqrt{x+4}$ dollars.

(a) Find a linear function $L(x)$ with $L(96) = C(96)$ and $L'(96) = C'(96)$.

(b) Use (a) to find an approximation to $C(100)$ (as usual, show your work).

$C(96) = 2\sqrt{100} = 20.$

7 $C'(x) = (x+4)^{-\frac{1}{2}}$. Thus, $C'(96) = \frac{1}{10}$.

(a) $L(x) = C(96) + C'(96)(x-96)$

$= 20 + \frac{1}{10}(x-96)$

3 (b) $C(100) \approx L(100) = 20.4$

(a) $L(x) =$ $20 + \frac{1}{10}(x-96) = (104+x)/10$

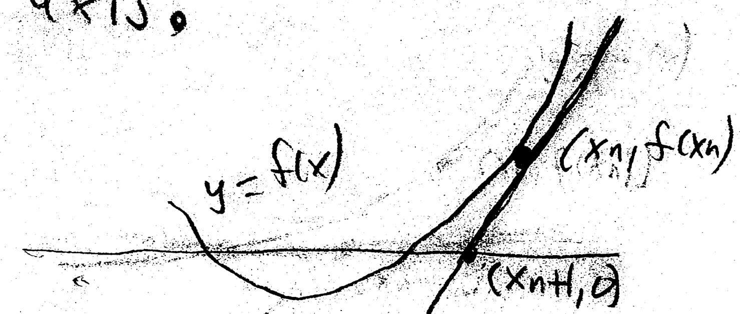
(b) $C(100) \approx$ $20.4 = 204/10$

(7) Given the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 1, 2, \ldots$ which is used in Newton's method.

(a) Briefly explain, using a diagram, how this formula is obtained. You do not need to actually derive the formula.

$(x_{n+1}, 0)$ is the intersection of the tangent line to $y = f(x)$ at $(x_n, f(x_n))$ with the x axis.

4



(b) Indicate how the above formula can be used to find the cube root of 9. In particular indicate (i) $f(x)$ and if $x_1 = 2$ find (ii) x_2.

6

$f(x) = x^3 - 9$, $f'(x) = 3x^2$

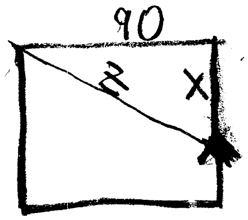
$X_{n+1} = X_n - \dfrac{(X_n^3 - 9)}{3X_n^2} = \frac{2}{3}X_n + \dfrac{3}{X_n^2}$. Thus,

using $X_1 = 2$, we get $X_2 = \frac{4}{3} + \frac{3}{4} = \frac{25}{12}$

(b) (i) $f(x) =$ $x^3 - 9$ (ii) $x_2 =$ $25/12$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) A baseball diamond is a square with 90 feet on a side. A batter hits the ball at home plate (one corner of the square) and runs toward first base with a speed of 24 feet per second (see diagram). At what rate is the runner's distance from second base decreasing when the runner is halfway to first base (you may give a decimal approximation as your answer)?



At time t let $x = x(t)$ be the distance of the runner from first base and z the distance of the runner from second base.

By the Pythagorean Theorem

4 $x^2 + 90^2 = z^2$. By implicit differentiation

4 $2x \frac{dx}{dt} = 2z \frac{dz}{dt}$. So $\frac{dz}{dt} = \frac{x}{z} \frac{dx}{dt}$

$x = 45$ when the runner is halfway to first,

3 $z^2 = 45^2 + 90^2$, $z = \sqrt{45^2 + 90^2}$

$= 45\sqrt{5}$.

3 Also $\frac{dx}{dt} = -24$ so $\frac{dz}{dt} = \frac{45}{45\sqrt{5}}(-24)$

$= \frac{-24}{\sqrt{5}}$ feet per second

Rate of decrease = $\frac{24}{\sqrt{5}}$ feet per second ≈ 10.733

(9) (a) State the chain rule. Use complete sentences.

4

If g is differentiable at x and f is differentiable at $g(x)$ then $f \circ g(x) = f(g(x))$ is differentiable at x and $\frac{d}{dx}(f \circ g)(x) = f'(g(x))\, g'(x)$. Equivalently if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

(b) Illustrate the chain rule by computing the derivative of $(i)\ h(x) = f(x^5)$ and $(ii)\ k(x) = (f(x))^5$ at $x = 1$ when $f(1) = 2$ and $f'(1) = -1$.

5

(i) $h'(x) = f'(x^5)\frac{d}{dx}(x^5) = 5x^4 f'(x^5)$.

Hence, $h'(1) = 5 f'(1) = -5$ when $x = 1$

5

(ii) $k'(x) = 5(f(x))^4 f'(x)$. Therefore

$k'(1) = 5(f(1))^4 f'(1)$

$= 5 \cdot 2^4 (-1) = -80$

$(i)\ h'(1) =$ -5 $(ii)\ k'(1) =$ -80

(10) Given the function, $f(x) = 3x^2 + 5x + 4$, whose graph is a parabola.

(a) Write the equation of the tangent line to this parabola at a point, $(a, 3a^2 + 5a + 4)$.

(b) Find all tangent lines to this parabola which pass through (1,0). Your solution must show how you found the tangent lines.

6 (a) Equation of tangent line is $y - f(a) = f'(a)(x-a)$

Now $f'(x) = 6x+5$ so $f'(a) = 6a+5$.
Put in the above equation to get

(∼) $y - (3a^2+5a+4) = (6a+5)(x-a)$ for the equation of the tangent line through $(a, f(a))$.

8 (b) When $y=0$, $x=1$ in (∼), so
$-(3a^2+5a+4) = (6a+5)(1-a) = -6a^2+a+5$.
Combine to get, $3a^2-6a-9 = 0$
or $3(a^2-2a-3) = 3(a-3)(a+1) = 0$
and $a = 3, -1$. Put these values into (∼) to conclude for $a = 3, -1$, that
$y = 23(x-1)$ and $y = -x+1$ are the tangent lines.

(a) $y - (3a^2+5a+4) = (6a+5)(x-a)$

(b) (i) $y = 23x - 23$ (ii) $y = -x+1$