MA 113 Spring 2009

Exam 2 - **KEY** March 10, 2009

1. a. Note that $\lim\limits_{t\to 0}\frac{\sin(5t^3)}{5t^3}=1$ and so rewriting and using the fact that the limit of a product is the product of the limits, we have:

$$\lim_{t\to 0} e^t\frac{\sin(5t^3)}{2t}=\lim_{t\to 0} e^t\frac{5t^2}{2}\frac{\sin(5t^3)}{5t^3}=\lim_{t\to 0} e^t*\lim_{t\to 0}\frac{5t^2}{2}*\lim_{t\to 0}\frac{\sin(5t^3)}{5t^3}=e^0*0*1=0$$

b. Again $\lim\limits_{x\to 0}\frac{4x}{\sin(4x)}=\lim\limits_{x\to 0}\frac{1}{\frac{\sin(4x)}{4x}}=\frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\frac{\sin(4x)}{4x}}=\frac{1}{1}=1$ and so, rewriting, we have:

$$\lim_{x\to 0} x\ln(x+e)\cot(4x)=\lim_{x\to 0}\frac{1}{4}*\frac{4x}{\sin(4x)}*\ln(x+e)*\cos(4x)$$

$$=\lim_{x\to 0}\frac{1}{4}*\lim_{x\to 0}\frac{4x}{\sin(4x)}*\lim_{x\to 0}\ln(x+e)*\lim_{x\to 0}\cos(4x)=\frac{1}{4}*1*\ln(e)*\cos(0)=\frac{1}{4}$$

2. a. $f'(x)=-\sin(4x)*4+2*\frac{1}{2}x^{-1/2}+\frac{1}{x}+0=-4\sin(4x)+\frac{1}{\sqrt{x}}+\frac{1}{x}$

b. $g'(x)=\frac{(x^4+2)*2x-x^2*(4x^3)}{(x^4+2)^2}=\frac{-2x^5+4x}{(x^4+2)^2}=\frac{-2x(x^4-2)}{(x^4+2)^2}$

c. $h'(x)=x^3*e^{5x^2+1}*10x+3x^2*e^{5x^2+1}=x^2(10x^2+3)e^{5x^2+1}$

3. a.
$$f'(x)=e^{x^2}*2x=2xe^{x^2}$$
$$f''(x)=2e^{x^2}+2x*2xe^{x^2}=2(1+2x^2)e^{x^2}$$

b. $g'(x)=\cos(\frac{\pi}{2}x)+(x-4)(-1)\sin(\frac{\pi}{2}x)*\frac{\pi}{2}$

$g''(x)=-\sin(\frac{\pi}{2}x)\frac{\pi}{2}-\sin(\frac{\pi}{2}x)\frac{\pi}{2}-(x-4)\cos(\frac{\pi}{2}x)(\frac{\pi}{2})^2$ *and since* $\sin(\frac{5\pi}{2})=1$ *and* $\cos(\frac{5\pi}{2})=0$ *we have* $g''(5)=-1*\frac{\pi}{2}-1*\frac{\pi}{2}-(5-4)*0*(\frac{\pi}{2})^2=-\pi$

4. a. $v(t) = s'(t) = 6t^2 - 30t + 36 = 6(t-2)(t-3)$ *and* $a(t) = s''(t) = 12t - 30$

b. Because the particle changes direction at t = 2 and t = 3, the total distance, T, traveled between t = 0 and t = 4 is given by:
$T = |s(2) - s(0)| + |s(3) - s(2)| + |s(4) - s(3)| = |29-1| + |28-29| + |33-28| = 28 + 1 + 5 = 34$ meters

5. Differentiating implicitly we have, $-y^2 + (3-x)*2*y*\frac{dy}{dx} = 2x$, and thus when x = 2 and y = 2 we have -4 + 1*2*2*$\frac{dy}{dx}$ = 2*2 so that the slope of the tangent line is $\frac{dy}{dx}\Big|_{\substack{x=2\\y=2}} = 2$ and thus the equation of the tangent line is y – 2 =2(x – 2) or y = 2x -2

6. a. Note that f(x) = ln(x+1) – ln(x^2+1) so that
$f'(x) = \frac{1}{x+1} - \frac{2x}{x^2+1}$ *and so* $f'(1) = \frac{1}{2} - \frac{2}{2} = -\frac{1}{2}$

b. Taking the ln of both sides we have
$\ln(g(x)) = \ln(e + \sin(x))^{\sin(x)} = \sin(x)\ln(e + \sin(x))$. Now differentiate and get
$\frac{g'(x)}{g(x)} = \cos(x)*\ln(e + \sin(x)) + \sin(x)*\frac{\cos(x)}{e + \sin(x)}$ and thus
$g'(\pi) = g(\pi)*(\cos(\pi)\ln(e + \sin(\pi)) + \sin(\pi)\frac{\cos(\pi)}{e + \sin(\pi)}) = e^0 * ((-1)*\ln(e) + 0*(\frac{-1}{e})) = -1$

7. We know that $m(t) = m_0 e^{kt}$ and thus $\frac{m(t)}{m_0} = e^{kt}$ so that, when t = 500 we have 20% of the substance left and therefore
$0.2 = \frac{m(500)}{m_0} = e^{500k} = (e^k)^{500}$ *so that* $e^k = (0.2)^{1/500}$ *and* $\frac{m(t)}{m_0} = (0.2)^{\frac{t}{500}}$. Letting t_1 be the half-life, we have $0.5 = \frac{m(t_1)}{m_0} = (0.2)^{\frac{t_1}{500}}$ *so that* $t_1 = 500*\frac{\ln(0.5)}{\ln(0.2)} = 215.34$ *years*

8. a. The slope of the tangent line at P(a, f(a)) is given by $f'(a) = 2a - 2$ so the equation of the tangent line is
$y - f(a) = (2a-2)(x-a)$ *but* $f(a) = a^2 - 2a + 2$ *so that* $y = (2a-2)(x-a) + a^2 - 2a + 2$
or $y = (2a-2)x + 2 - a^2$

b. We want the tangent lines that go through the point P(3, 4). That is we want the values of a so that when x = 3 then y = 4. That is:

4 = (2a-2)*3 + 2 – a^2 or a^2 – 6a + 8 = 0 = (a – 2)(a – 4) so that a = 2 or a = 4. Thus the two tangent lines that go through P(3, 4) are y = 2x – 2 and y = 6x – 14.

9. a. If f = h ∘ g, h is differentiable at g(x) and g is differentiable at x, then f is differentiable at x and $f'(x) = h'(g(x)) * g'(x)$.

b. (i) $h'(x) = \frac{2f'(x)}{f(x)}$ so $f'(2) = \frac{2f'(2)}{f(2)} = \frac{2*(-1)}{3} = -\frac{2}{3}$

(ii) $l'(x) = f'(x^3 g(x)) * (x^3 g'(x) + 3x^2 g(x))$ and so

$l'(2) = f'(2^3 g(2))(2^3 g'(2) + 3*2^2 g(2)) = f'(8*\frac{1}{4})(8*2+3*4*\frac{1}{4}) = 19f'(2) = 19*(-1) = -19$

10. Let y = y(t) = the length of rope between the boat and the pulley at time t and x = x(t) = the horizontal distance between the boat and the dock at time t. We are given that $y'(t) = -4\, ft./\text{sec}$. and asked to find $x'(t)$ when x = 12ft. By the Pythagorean Theorem we have that $y^2 = x^2 + 5^2$ and differentiating we get $2yy' = 2xx'$ or $x' = \frac{yy'}{x}$. Now, when x = 12, $y^2 = 12^2 + 5^2 = 13^2$ so y = 13 and thus, when x = 12, $x' = \frac{13*(-4)}{12} = -\frac{13}{3}$ft./sec.