Calculus I | Common exam
Exam 3 | 14 November 2006

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*.

Each question is followed by space to write your answer. Please lay out your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question.

You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name Solutions & Grading Guide

Section ____________

Last four digits of student identification number ____________

Assignments

Weaver \ Mattingly pg 1
Miker \ Grogan pg 2
Zeckner \ Merrick pg 3
Simon \ Taylor pg 4
Boucher \ Keach pg 5
Perry \ Little Q10
Martin \ Brown Q11
Harris \ Frayer Q12

Question	Score	Total
Q1,2/p. 1		14
Q3,4/p. 2		14
Q5/p. 3		14
Q6,7/p. 4		14
Q8,9/p. 5		14
Q10		14
Q11		14
Q12		14
Free	2	2
		100

9 pts

1. Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 6x^2 + 9x + 6$ on the interval $[0, 5/2]$.

2pts. $f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1)$

Solve $f'(x) = 0$ $x=3$ and $x=1$.

But $x = 3$ is not in $[0, 5/2]$.

2 pts: Critical Numbers $\{0, 1, 5/2\}$ in $[0, 5/2)$

Compute Values: $f(0) = 6$

3pts. $f(1) = 10$

$f(5/2) = 6\ 5/8$

2pts Abs. Max 10 at $x=1$

Abs. Min 6 at $x=0$

Absolute minimum value 6., Absolute maximum value 10

2. Let f be a function which is defined for all real numbers and so that f and its first and second derivatives are continuous.

5pts. Which of the following statements must be true for all such f? Circle T if the statement is true for all such f. Circle F if there is at least one such function for which the statement is

1 pt. each not true.

No justification is needed.

(a) T F If f has a local maximum at 42, then $f'(42) = 0$.

(b) T F If $f'(a) = 0$, then f has a local maximum or minimum at a.

(c) T F If the absolute maximum for f on the closed interval $[1, 2]$ is at $x = 1$, then $f'(1) = 0$.

(d) T F If $f(1) = f(3)$, then $f'(2) = 0$.

(e) T F If the second derivative f'' has $f''(x) > 0$ for all x, then the first derivative f' is increasing on the interval $(-\infty, \infty)$.

3. Suppose that

$$f(x) = \frac{x^A + 10}{x^2 - B}.$$

7 pts

Find values A and B so that this function has horizontal asymptote at $y = 1$ and a vertical asymptote at $x = 3$.

For these values of A and B, the graph of the function will have one more vertical asymptote besides $x = 3$. Find this asymptote.

Horizontal Asymp. $\lim_{x \to \infty} \frac{x^A\left(1 + \frac{10}{x^A}\right)}{x^2\left(1 - \frac{B}{x^2}\right)} = 1$ requires $A = 2$

3pts

(so can also take $x \to -\infty$)

Vertical Asymp. at $x = 3$ $3^2 - B = 0$ so $\frac{x^2+10}{x^2-9} = \frac{x^2+10}{(x-3)(x+3)}$ Vertical As at $x = 3$ and $x = -$ (even fnc

3pts $\lim_{x \to 3^+} f(x) = +\infty$ and $\lim_{x \to +3^-} f(x) = -\infty$.

$A =$ 2, $B =$ 9,

1pt. Vertical asymptote at $x =$ -3.

7 pts 4. Let $f(x) = 4x^3 - 24x^2 - 540x$. Find the interval(s) where the graph of $y = f(x)$ is concave up and the interval(s) where the graph is concave down. Also determine the inflection point(s).

Compute: $f'(x) = 12x^2 - 48x - 540$

2pts. $f''(x) = 24x - 48 = 24(x-2)$

2pts $f'' > 0$ $x > 2$ or $(2, \infty)$

3pts $f'' < 0$ $x < 2$ or $(-\infty, 2)$

Inflection point(s) at $x =$ 2 because concavity changes from up to down

Interval(s) where f is concave up $x > 2$ or $(2, \infty)$ $(f'' > 0)$

Interval(s) where f is concave down $x < 2$ $(f'' < 0)$ or $(-\infty, 2)$

6. Let $f(x) = x\sin(x)$.

7 pts (a) Write an equation whose solutions are the critical numbers for f.

(b) Carry out one step of Newton's method to solve the equation you wrote in part (a). Let $x_1 = \pi/2$ and give an exact value for x_2.

3 pts a) $f'(x) = \sin x + x\cos x = 0$ gives the critical numbe[r]

4 pts b) $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$, $f''(x) = 2\cos x - x\sin x$

2 pts $x_1 = \pi/2 \Rightarrow f'(\pi/2) = 1$

$f''(\pi/2) = -\pi/2$

2 pts so $x_2 = \pi/2 - \frac{1}{(-\pi/2)} = \frac{\pi}{2} + \frac{2}{\pi}$

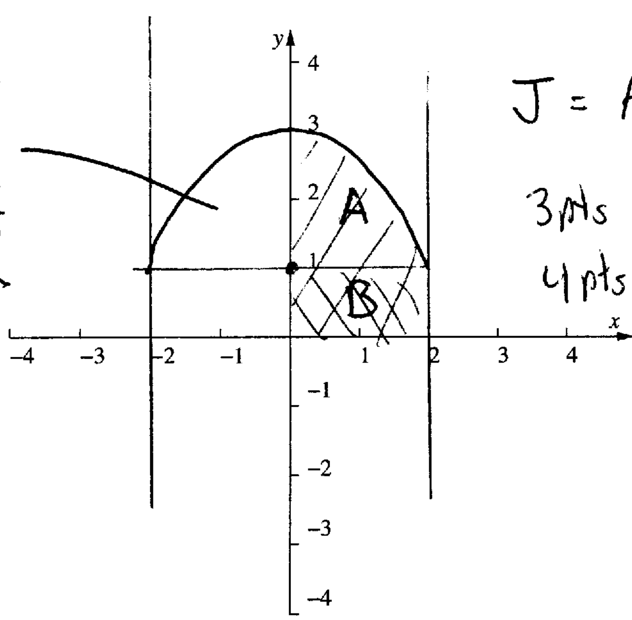
(a) $\sin x + x\cos x = 0$, (b) $x_2 =$ $\frac{\pi}{2} + \frac{2}{\pi}$

7. Sketch the graph of the function $f(x) = 1 + \sqrt{4 - x^2}$. Use a geometric argument to find the exact value of the integral

7 pts

$$J = \int_0^2 (1 + \sqrt{4 - x^2})\, dx.$$

top half of the circle of radius 2 centered at $(0,1)$



$J = A + B = \frac{1}{4}\pi(2)^2 + (2 \cdot 1)$

3 pts graph $= \pi + 2$.

4 pts calculation

$4 - x^2 = y^2$ circle centered at $(0,0)$ of radius 2. We shift this up by 1.

$\int_0^2 (1 + \sqrt{4 - x^2})\, dx =$ $\pi + 2$.

8. Consider the integral $\int_0^{\pi} 3\sin(x)\,dx$.

7 pts

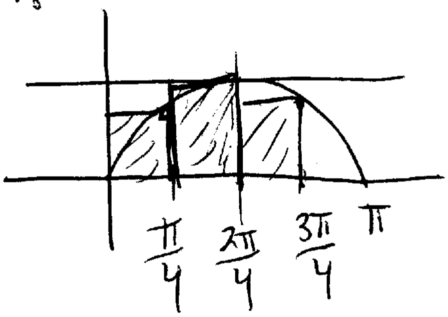
Compute the Riemann sum for this integral which is obtained by partitioning the interval $[0, \pi]$ into four sub-intervals of equal width and using the right endpoint for each interval as the sample point.

$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$

1 pt.

Right endpt. $\int_0^{\pi} 3\sin x\,dx = 3\int_0^{\pi} \sin$

1 pt.



$R_4 = 3\cdot\frac{\pi}{4}\left(f(\pi/4) + f(2\pi/4)\right) + f(3$

3 pts

since $f(\pi) = 0$, $f(x) = \sin x$

2 pts.

$R_4 = \frac{3\pi}{4}\left(\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right)$

$= \frac{3\pi}{4}(\sqrt{2} + 1)$

7 pts

Value of Riemann sum $\frac{3\pi}{4}(\sqrt{2}+1) \approx 5.6884$

9. Find a function p which satisfies $p'(x) = \sin(x) + 2\cos(x)$ and $p(0) = 2$.

4 pts Antiderivative: $p(x) = -\cos x + 2\sin x + C$

Initial Condition $p(0) = -1 + C = 2$

$C = 3$

3 pts.

$p(x) =$ $p(x) = -\cos x + 2\sin x + 3$

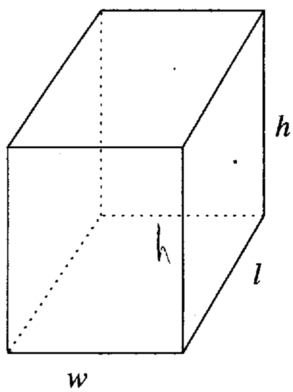
Answer two of the following three questions. Indicate the question that is not to be graded by marking through this question on the front of the exam.

10. A box has a rectangular base, four rectangular sides, but no top. The length of the base is twice the width of the base. The volume of the box is 288 cubic centimeters.

    (a) Express the total outer area of the sides and base as a function of w, the shorter dimension of the base.

    (b) Find the dimensions of the box which has least surface area.

    (c) Explain how you know you have found the absolute minimum.



a) Area = area of sides + area of bottom

7pts.

Area $= 2hl + 2hw + lw$ 2pts

Constraint 1: $l = 2w$

so

$$A(h, w) = 6hw + 2w^2$$

Constraint 2: $V = wlh = 288 \text{ cm}^3$ 2pts

$= 2w^2 h = 288$ so $h = \frac{144}{w^2}$ cm

Finally: $\boxed{A(w) = \frac{864}{w} + 2w^2, \quad w > 0}$ 3pts

7pts

b) Minimize $A(w)$: $A'(w) = -\frac{864}{w^2} + 4w = 0$

3pts

$w^3 = 216$ 2pts

$w = \sqrt[3]{216}$ cm $= 6$ cm

$\boxed{\begin{array}{l} l = 12 \text{ cm} \\ w = 6 \text{ cm} \\ h = \frac{144}{36} = \end{array}}$

(c) Global min $\left\{ \begin{array}{l} A''(w) = 4 + \frac{928}{w^3} > 0 \text{ Concave down} \\ \text{so global minimum. 2pts.} \end{array} \right.$

or note $A'(w) > 0$ $w > 6$ and $A'(w) < 0$ if $0 < w < 6$

11. (a) State the mean value theorem.

14 pts. (b) Find a function so that $f'(x) \leq 2$, $f(0) = 2006$ and $f(2) = 2010$.

(c) Suppose that f is a function which is differentiable for all x, satisfies $-1 \leq f'(x) \leq 2$ for all x and $f(0) = 3$. Use the mean value theorem to show that that $f(x) \leq 2x + 3$ for $x \geq 0$.

a) If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a point c in (a,b)

6 pts. So that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

b) Compute $\dfrac{f(2)-f(0)}{2-0} = \dfrac{2010-2006}{2} = \dfrac{4}{2} = 2$.

3 pts. Try the straight line:

$$\frac{f(x) - 2006}{x} = 2 \quad \text{or} \quad f(x) = 2006 + 2x$$

Check: $f'(x) = 2$
$f(2) = 2010$

c) Use MVT on $[0,x]$, $x \geq 0$. There is a c in $(0,x)$

5 pts. $f'(c) = \dfrac{f(x) - f(0)}{x} \leq 2$ since $f'(x) \leq 2$ for a

This implies: $f(x) \leq f(0) + 2x = 3 + 2x$ for $x \geq$

12. Let $f(x) = (x-1) + \frac{1}{x-1}$.

14 pts.

(a) Find the y-intercept(s) of the graph of f.

(b) Find all vertical asymptotes to the graph of f.

(c) Compute $f'(x)$ and give the domain of $f'(x)$.

(d) Use the first derivative to determine the intervals of increase and decrease for f and find all local maxima and local minima for f.

(e) Compute $f''(x)$ and give the domain of $f''(x)$.

(f) Use the second derivative to find intervals of concavity for f.

(g) Sketch the graph of f and label all local extrema. Sketch the vertical asymptote(s) with dashed lines. (Use the axes on the next page.)

1pt. a) $f(0) = -2$ y-intercept is $\underline{y = -2}$ (only one)

1pt b.) Vertical Asympt. is $\underline{x = 1}$ since $\lim_{x \to 1^+} f(x) = +\infty$

(also can use $\lim_{x \to 1^-} f(x) = -\infty$)

2pts. c) $f'(x) = 1 - \frac{1}{(x-1)^2}$ Domain is $\{x \mid x \neq 1\}$ or $(-\infty, 1) \cup (1, \infty)$

4pts. d) $f'(x) = 0$ at $1 = \frac{1}{(x-1)^2}$ or $(x-1)^2 = 1$

$x^2 - 2x = 0$

$x(x-2)$ so $\underline{x = 0 \text{ or } 2}$

	$x<0$	$0<x<1$	$1<x<2$	$x>2$
$f'(x)$	+	−	−	+
f	incr	decr	decr	Incr

max min

local max at $x = 0$

local Min at $x = 2$.

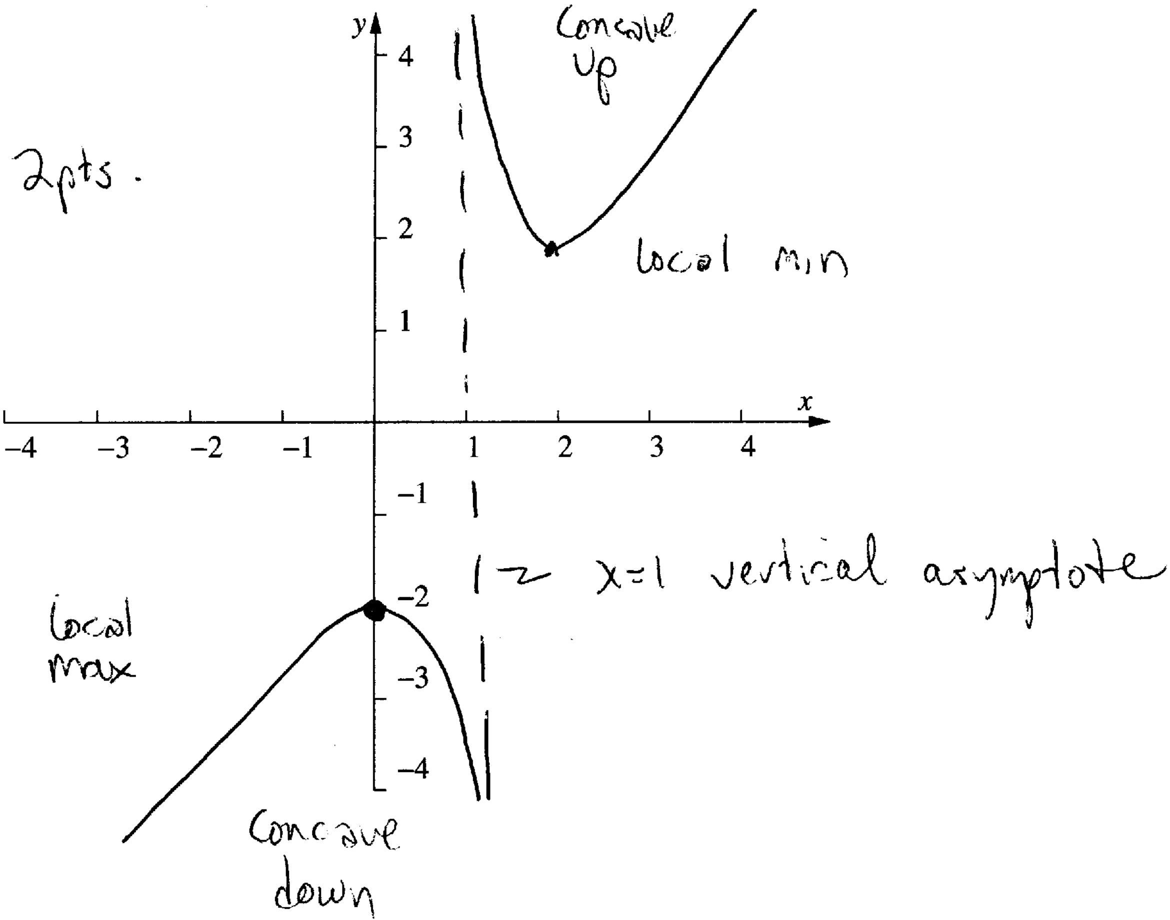
$f(0) = -2$ $f(2) = 2$

2pts e) $f''(x) = \frac{2}{(x-1)^3}$ Domain is $\{x \mid x \neq 1\}$ or $(-\infty, 1) \cup (1$

f) $f''(x) > 0$ $x > 1$ Concave up on $(1, \infty)$

2pts $f'' < 0$ $x < 1$ Concave down on $(-\infty, 1)$

2pts.


y
x
4
3
2
1
−1
−2
−3
−4
−4
−3
−2
−1
1
2
3
4
Concave Up
local min
local max
Concave down
x=1 vertical asymptote