Exam 3

Solutions

Multiple Choice Questions

1. Suppose that $\dfrac{dy}{dt} = ky$, where k is a constant and suppose that $y(0) = 2$ and $y(2) = 6$. Find $y(5)$.

   A. $2 \cdot 6^5$

   B. $2 \cdot 6^{5/2}$

   C. $2 \cdot 3^5$

   D. $2 \cdot 3^{5/2}$

   E. None of the above.

Solution: Since the differential equation is of the form $\dfrac{dy}{dt} = ky$, where k is a constant, we know that the solution is $y(t) = C_0 e^{kt}$. Using the fact that $y(0) = 2$ gives us easily that $C_0 = 2$. Thus we have that $y(t) = 2e^{kt}$. Now, use the fact that $y(2) = 6$, we have that

$$\begin{aligned} 6 = y(2) &= 2e^{2k} \\ 3 &= e^{2k} \\ k &= \frac{\ln 3}{2} \end{aligned}$$

Thus,

$$y(t) = 2e^{(\ln 3)\frac{t}{2}} = 2 \cdot 3^{t/2}.$$

Therefore, $y(5) = 2 \cdot 3^{5/2}$.

2. The half-life of a radioactive substance is 20 years. If a sample has a mass of 100 grams, how much of the sample remains after 50 years?

   A. $100 \cdot e^{\frac{5\ln(2)}{2}}$ grams

   B. $100 \cdot e^{-\ln(5/2)}$ grams

   C. $100 \cdot e^{-5/2}$ grams

   D. $100 \cdot e^{-\frac{5\ln(2)}{2}}$ grams

   E. 12.5 grams

Solution: We know that the function $f(t) = 100e^{kt}$ is the function that describes the amount of the substance present after t years. In order to find k we will use the half-life.

$$\begin{aligned} 50 &= 100e^{20k} \\ e^{20k} &= \frac{1}{2} \\ 20k &= -\ln 2 \\ k &= -\frac{\ln 2}{20} \end{aligned}$$

Thus, $f(t) = 100 \cdot e^{-(t\ln 2)/20}$ so after 50 years there are $f(50) = 100 \cdot e^{-\frac{5\ln(2)}{2}}$ grams remaining.

3. Find the rate of change of the volume of a cube with respect to the length of its side s when $s = 9$ meters.

   A. 729 cubic meters of volume per meter of length

   B. 243 cubic meters of volume per meter of length

   C. 81 cubic meters of volume per meter of length

   D. 9 cubic meters of volume per meter of length

   E. None of the above.

Solution: The volume of a cube is $V = s^3$ thus, the rate of change of volume with respect to s is $\frac{dV}{ds} = 3s^2$, so at $s = 9$ this rate of change is $3 \times 9^2 = 243$ cubic meters of volume per meter of length.

4. Two cars leave Lexington at the same time, one traveling due east at 60 miles per hour and one traveling due south at 75 miles per hour. How fast is the distance between them changing after 2 hours?

   A. $\approx$ 192 mph

   B. $\approx$ 137 mph

   C. $\approx$ 68 mph

   D. $\approx$ 96 mph

   E. $\approx$ 135 mph

Solution: Let x denote the distance that the car traveling due east has traveled at time t. Let y denote the distance that the car traveling due south has traveled at time t. We know that

$$\frac{dx}{dt} = 60 \text{ mph and } \frac{dy}{dt} = 75 \text{ mph}.$$

The distance, z, between the cars is given by $z^2 = x^2 + y^2$. We need to find $\frac{dz}{dt}$ when $t = 2$. Note that when $t = 2$, $z^2 = 120^2 + 150^2 = 36900$ or $z = 30\sqrt{41}$.

$$\begin{aligned}
z^2 &= x^2 + y^2 \\
2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{2y}{dt} \\
\frac{dz}{dt} &= \frac{x\frac{dx}{dt} + y\frac{2y}{dt}}{z} \\
&= \frac{60 \cdot 120 + 75 \cdot 150}{30\sqrt{41}} \\
&= \frac{615}{\sqrt{41}} \approx 96.047 \text{ mph}
\end{aligned}$$

5. Find the average rate of change of the area of a circle with respect to its radius r as r changes from 3 to 8.

   A. 6π

   B. 8π

   C. 11π

   D. 12π

   E. 36π

Solution: The average rate of change of the area with respect to its radius is the difference in the values of the area divided by the difference in the radii.

$$\frac{\pi(8)^2 - \pi(3)^2}{8-3} = \frac{55\pi}{5} = 11\pi.$$

6. Find the critical points of $f(x) = \dfrac{x^2}{x^2 - 4x + 9}$.

   A. $\{0, -9/2\}$

   B. $\{0, 9/2\}$

   C. $\{0, 3/2\}$

   D. $\{0, 9/4\}$

   E. None of the above.

Solution: We need to find where the derivative is zero.

$$f'(x) = \frac{(x^2-4x+9)(2x) - (x^2)(2x-4)}{(x^2-4x+9)^2} = \frac{18x - 4x^2}{(x^2-4x+9)^2}$$

This is zero when the numerator is zero, so setting $18x - 4x^2 = 0$, we get that the critical points are $x = 0$ and $x = 9/2$.

7. Suppose that $f'(x) = x^2(x+2)(x-2)(x-4)$. Find the interval or intervals where f is decreasing. (Read the problem carefully. The given function is $f'(x)$, not $f(x)$.)

   A. $(-\infty, -2) \cup (2, \infty)$

   B. $(-2, 2) \cup (4, \infty)$

   C. $(-\infty, -2) \cup (2, 4)$

   D. $(2, \infty)$

   E. $(2, 4)$

Solution: Since we are given the derivative, $f'(x)$, we just need to see where it is negative to find where the function $f(x)$ is decreasing. Partitioning the real line by the zeroes of the derivative we divide the real line at the points -2, 0, 2, and 4. Checking each of the intervals we find that the derivative is negative for $x \in (-\infty, -2) \cup (2, 4)$.

8. You are given that $f'(x) = x^2(x+2)(x-2)(x-4)$. Find the values of x that give the local maximum and local minimum values of the function $f(x)$. (Read the problem carefully. The given function is $f'(x)$, not $f(x)$.)

   A. Local maximum value of f at $x = 0$ and local minimum values of f at $x = -2, 4$

   B. Local maximum values of f at $x = -2, 4$ and local minimum value of f at $x = 0$

   C. **Local maximum value of f at $x = 2$ and local minimum values of f at $x = -2, 4$**

   D. Local maximum values of f at $x = -2, 2$ and local minimum values of f at $x = 0, 4$

   E. Local maximum values of f at $x = 0, 4$ and local minimum values of f at $x = -2, 2$

Solution: Again we are given the derivative. Reading from the derivative, we see that the function is decreasing for $x < -2$, increasing for $-2 < x < 2$, decreasing for $2 < x < 4$ and then increasing for $x > 4$. From this and the First Derivative Test we see that the function has a local maximum at $x = 2$ and local minima at $x = -2, 4$.

9. Assume that $f''(x) = x(x-2)(x-4)$. Find the points of inflection of the function f. (Read the problem carefully. The given function is $f''(x)$, not $f(x)$.)

   A. $x = 0, 2, 4$

   B. $x = 2$

   C. $x = 4$

   D. $x = 0, 4$

   E. $x = 2, 4$

Solution: The function has inflection points where it changes concavity; that is, where the second derivative changes sign. The second derivative changes sign at $x = 0$, 2 and 4, so these three points are the x coordinates of the inflection points for $f(x)$.

10. The function $f(x) = e^{|x|}$ has an absolute minimum at $x = 0$ because:

    A. $f'(0) = 0$ and $f''(0) > 0$.

    B. $f'(x) > 0$ **for** $x > 0$ **and** $f'(x) < 0$ **for** $x < 0$**, with** $f'(0)$ **undefined.**

    C. $f(x)$ is not differentiable at $x = 0$ and $f''(0) > 0$.

    D. this is the statement of the Mean Value Theorem.

    E. None of the above.

Solution: The function has an absolute minimum at $x = 0$ because $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$, with $f'(0)$ undefined.

11. Find all values c that satisfy the conditions of the Mean Value Theorem for

$$f(x) = 7x^2 - x + 5$$

on the interval $[a, b] = [-1, 7]$.

A. 3

B. 4

C. $2, 3$

D. $2, 4$

E. None of the above

Solution: We need to find the slope of the secant line on the interval $[-1, 7]$.

$$\frac{f(7) - f(-1)}{7 - (-1)} = \frac{328}{8} = 41.$$

Now, we need to find a point in $(-1, 7)$ where the derivative equals 41.

$$\begin{aligned} f'(x) &= 41 \\ 14x - 1 &= 41 \\ x &= 3 \end{aligned}$$

12. Find two positive numbers whose product is 144 and whose sum is a minimum.

A. 2, 72

B. 3, 48

C. 4, 36

D. 6, 24

E. None of the above

Solution: Let the numbers be x and y. We are given that $x > 0$, $y > 0$ and $xy = 144$. We want to minimize the sum $s = x + y$. Solve for y in terms of x from our given information giving us $y = 144/x$. Then we must minimize the function $s = x + \frac{144}{x}$. The derivative is $\frac{ds}{dx} = 1 - \frac{144}{x^2}$. Setting this equal to 0 gives us that $x = \pm 12$. Since $x > 0$, the critical point here is $x = 12$. We can check the concavity to see if this is a minimum. $\frac{d^2s}{dx^2} = \frac{288}{x^3}$ which is positive for x positive. Thus, the function is concave up for all $x > 0$ which gives us that $x = 12$ gives a global minimum. Solving for y now, gives us that $y = 12$ also and the minimum sum is $12 + 12 = 24$.

Free Response Questions
Show all of your work

13. Find the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\sin(2x)}$

Solution: $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\sin(2x)} = \frac{0}{0}$ so we can use l'Hospital's Rule and

$$\lim_{x\to 0} \frac{e^x - 1}{\sin(2x)} = \lim_{x\to 0} \frac{e^x}{2\cos(2x)} = \frac{1}{2}$$

(b) $\displaystyle\lim_{x\to 0} \frac{1 - \cos(x)}{x^4 + 3x^2}$

Solution: Here we will have to use l'Hospital's Rule twice.

$$\lim_{x\to 0} \frac{1 - \cos(x)}{x^4 + 3x^2} = \lim_{x\to 0} \frac{\sin x}{4x^3 + 6x} = \lim_{x\to 0} \cos x 12x^2 + 6 = \frac{1}{6}$$

(c) $\displaystyle\lim_{x\to 0} \frac{e^{3x} - e^{-9x}}{\ln(1 + x)}$

Solution: Here we only have to use l'Hospital's Rule once.

$$\lim_{x\to 0} \frac{e^{3x} - e^{-9x}}{\ln(1 + x)} = \lim_{x\to 0} \frac{3e^{3x} + 9e^{-9x}}{\dfrac{1}{1 + x}} = 12$$

14. The volume of a right circular cone of radius r and height h is $V = \frac{\pi}{3} r^2 h$. Suppose that the radius and height of the cone are changing with respect to time t.

(a) Find a relationship between $\frac{dV}{dt}$, $\frac{dr}{dt}$, and $\frac{dh}{dt}$.

Solution:

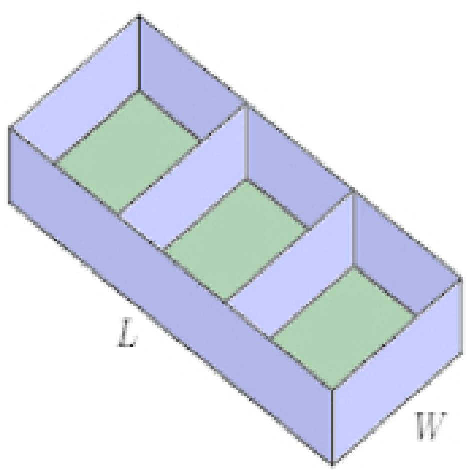
$$V = \frac{\pi}{3} r^2 h$$
$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

(b) At a certain instant of time, the radius and height of the cone are 12 in. and 13 in. and are increasing at the rate of 0.2 in/sec and 0.5 in/sec, respectively. How fast is the volume of the cone increasing?

Solution: Here we only need to substitute the given values into the above equation:

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2 \cdot 12 \cdot 13 \cdot 0.2 + 12^2 \cdot 0.5\right) = 44.8\pi \text{ cubic in/sec}$$

15. Your task is to design a rectangular industrial warehouse consisting of three separate spaces of equal size.



The wall materials cost \$66 per linear foot and your company has allocated \$79200 for those walls.

(a) Find the dimensions which use all of the budget and maximize the total area.

Solution: Since the materials are \$66 per linear foot and you are allocated \$79200, the maximum amount of wall you can use is 79200/66=1200 feet. The total linear footage of the walls is $2L + 4W$ so we must have that $2L + 4W = 1200$ and we want to maximize the area which is $A = LW$. We can solve for L in terms of W getting $L = 600 - 2W$. Plugging this into the area function gives

$$\begin{aligned} A &= LW \\ &= W(600 - 2W) = 600W - 2W^2 \\ A' &= 600 - 4W \end{aligned}$$

Setting $A' = 0$ we get one critical point, $W = 150$. Checking $A'' = -4$ means that A is concave down for all W so $W = 150$ gives us the global maximum. If $W = 150$, $L = 600 - 2W = 300$ and the warehouse is to be 150 feet by 300 feet.

(b) What is the area of the three (equal size) compartments?

Solution: The area of the warehouse is 45000 square feet, so each compartment has an area of 15,000 square feet.

16. Let $f(x) = x^4 - 18x^2 - 7$. Be sure to justify each of your answers below.

(a) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

Solution: We need to find where the derivative is positive and where it is negative.

$$f'(x) = 4x^3 - 36x = 4x(x^2 - 9) = 4x(x + 3)(x - 3)$$

The critical points are then $x = -3$, $x = 0$ and $x = 3$. These divide the real line into four intervals. Checking a number in each of the intervals gives us that $f(x)$ is increasing for $x \in (-3, 0) \cup (3, \infty)$ and decreasing for $x \in (-\infty, -3) \cup (0, 3)$.

(b) Find the intervals where $f(x)$ is concave up and the intervals where $f(x)$ is concave down.

Solution: Here we need the second derivative and its zeros.

$$f''(x) = 12x^2 - 36 = 12(x^2 - 3)$$

. This is zero when $x = \pm\sqrt{3}$. These two points divide the real line into three regions and we see that $f(x)$ is concave up for $x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ and concave down for $x \in (-\sqrt{3}, \sqrt{3})$.

(c) Find the points that give local maximum values of $f(x)$, the points that give local minimum values of $f(x)$, and the points of inflection of $f(x)$.

Solution: Using the Second Derivative Test, we have that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = -3, 3$. The points of inflection are at $x = \pm\sqrt{3}$ since the concavity changes there.