Name: ______________________ Section and/or TA: ______________

Do not remove this answer page. You will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or communication capabilities is permitted. You may not use a phone or other communication device during the exam.

The exam consists of 12 multiple choice questions that are worth 5 points each and 4 free response questions that are worth 10 points each. You should work the multiple choice questions on the question page. After you have checked your work carefully, record your answers by completely filling in the circle below that corresponds to your answer. If you must change your answer, make a note on the front of the exam. Be sure to check carefully when you transfer your answers to the cover sheet.

Show all work to receive full credit on the free response problems.

Multiple Choice Questions

1 A B C D E

2 A B C D E

3 A B C D E

4 A B C D E

5 A B C D E

6 A B C D E

7 A B C D E

8 A B C D E

9 A B C D E

10 A B C D E

11 A B C D E

12 A B C D E

SCORE

Multiple Choice	13	14	15	16	Total Score
60	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Suppose that at a certain time, $r = 2$ centimeters, and the radius is increasing at rate of 3 centimeters/second. Find the rate of change of the volume at this time.

   A. 48π cubic centimeters per second

   B. 36π cubic centimeters per second

   C. 54π cubic centimeters per second

   D. 32π cubic centimeters per second

   E. 72π cubic centimeters per second

Solution: We have $V' = 4\pi r^2 r'$. Setting $r = 2$ m and $r' = 3$ m/s, we have $V' = 4\pi(2)^2 \cdot 3 = 48\pi$ cubic centimeters per second.

Compare WW 3.2 #1

2. (5 points) Suppose a population at time t consists of $P(t) = 20e^{0.06t}$ individuals. Find the time it takes for the population to triple.

   A. $1/0.02$

   B. $\ln(3)/0.06$

   C. $\ln(0.06)/3$

   D. $20/3$

   E. 60

Solution: Compare WW 3.3 #2

3. (5 points) Let $f(x) = \dfrac{1}{1+x}$. Find the quadratic approximation to f at $x = 0$.

   A. $1 - x - \dfrac{x^2}{2}$

   B. $1 - x + x^2$

   C. $1 - x - x^2$

   D. $1 + x + x^2$

   E. $-1/4$

4. (5 points) For which value of x does $f(x) = x^5 - x^3$ have a local maximum at x?

   A. $-\sqrt{3/5}$

   B. 0

   C. $\sqrt{3/10}$

   D. $\sqrt{3/5}$

   E. $-\sqrt{3/10}$

Solution: Compare WS 3.5.1–2, #3

5. (5 points) Let $f(x) = x^2 - 8x + 3$ on the interval $[3, 7]$. Find the value c so that the global (or absolute) maximum value for f on the interval $[3, 7]$ is at c.

   A. 4

   B. 7

   C. 6

   D. 3

   E. 5

Solution: The derivative is $f'(x) = 2x - 8$. We have a critical number at $x = 4$ since $f'(4) = 0$. Testing the critical number and endpoints, we find

x	3	4	7
$f(x)$	-12	-13	-4

The largest value is at $x = 7$.

6. (5 points) The *derivative* of f is $f'(x) = x(x-2)(x-3)$. Find the interval or intervals where f is decreasing.

   A. $(0, 2)$ and $(3, \infty)$

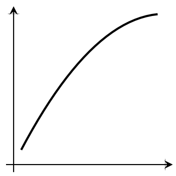
   B. $(3, \infty)$

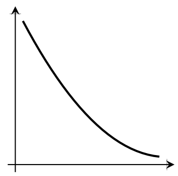
   C. $(-\infty, 0)$ and $(3, \infty)$

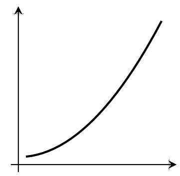
   D. $(0, 2)$

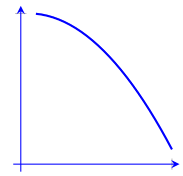
   E. $(-\infty, 0)$ and $(2, 3)$

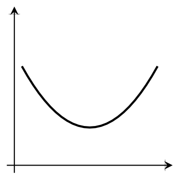
7. (5 points) Select the graph of the function which is decreasing and concave down.

A. 

B. 

C. 

D. 

E. 

Solution: Compare WS 3.6.3 #2, WW 3.6.3, #1

8. (5 points) Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{3x^2+1}{2x-3}$

A. Vertical asymptote at $x = 2/3$, horizontal asymptote at $y = 3/2$

B. Vertical asymptote at $x = 3/2$, horizontal asymptote at $y = 2/3$

C. Vertical asymptote at $x = 3/2$, horizontal asymptote at $y = 0$

D. Vertical asymptote at $x = 3/2$, no horizontal asymptote

E. Vertical asymptote at $x = 3/2$, horizontal asymptote at $y = 3/2$

Solution: Compare WS 3.6.1-2, # 4.

9. (5 points) Find the limit $\lim_{x \to 0} \dfrac{\sin(4x)}{\sin(2x)}$.

   A. 4

   B. 1/2

   C. 2

   D. 1/4

   E. Does not exist

Solution:

10. (5 points) Consider the limit $\lim_{x \to 1} \dfrac{x^2 + \sin(\pi x) - K}{x - 1}$. Find the value of K so that the limit exists and is a finite number.

   A. π

   B. 1

   C. 0

   D. 2

   E. $\pi + 1$

Solution: Compare WW 3.7 #6.

11. (5 points) Find the general anti-derivative of $f(x) = x + \dfrac{1}{1+x^2}$.

A. $1 + \dfrac{-2x}{1+x^2} + C$

B. $2x^2 + \ln(1+x^2) + C$

C. $\dfrac{x^2}{2} + \arctan(x) + C$

D. $\dfrac{x^2}{2} + \ln(1+x^2) + C$

E. $2x^2 + \arctan(x) + C$

12. (5 points) Let $f(x) = \sin(x) + \cos(x)$. Find F, an anti-derivative of f with $F(0) = 2$.

A. $F(x) = 3 - \sin(x) - \cos(x)$

B. $F(x) = 1 - \sin(x) + \cos(x)$

C. $F(x) = 1 + \sin(x) + \cos(x)$

D. $F(x) = 2 - \sin(x) + \cos(x)$

E. $F(x) = 3 + \sin(x) - \cos(x)$

Solution: Compare WS Review 3 #5.

Free response questions, show all work

13. (10 points) The half-life of a radioactive substance is 6000 years. This means that its mass will decrease to half its original value after 6000 years.
    (a) Find the decay rate k so that $M(t) = M_0 e^{-kt}$ gives the mass of M after t years.
    (b) Suppose that $M(T) = \frac{1}{10} M(0)$. Find T.

Solution: Compare WS 3.3 # 3, Review WS #2.

a) We want to find k so that $e^{-k6000} = 1/2$.

Applying the function ln, we have $-k6000 = \ln(1/2) = -\ln(2)$ and then $k = \ln(2)/6000$. Using a calculator $k \approx 1.1552 \times 10^{-4} = 0.00011552\,\text{years}^{-1}$.

b) We need $e^{-kT} = 1/10$ and applying the natural logarithm, we obtain the equation $-kT = \ln(1/10) = -\ln(10)$. Solving for T, gives $T = 6000 \ln(10)/\ln(2)$. As a decimal $T \approx 19,932$ years.

Grading. Exponential equation for k, $e^{-k6000} = 1/2$. (3 points). Result of applying ln (1 point). Value for k (1 point). Accept either $\ln(2)/6000$ or $-\ln(1/2)/6000$.

Decimal answer is not required (or very useful). Units are not expected for k, but give a gold $\star$ to students who have that right.

b) Equation for T, $e^{-kT} = 1/10$ (1 point). Apply ln correctly (1 point). Solve for T (1 point) and substitute value for k (1 point). Units for T (1 point).

14. (10 points) A 4 meter ladder leans against a wall. The bottom of the ladder is sliding away from the wall at 0.3 meter/second. When the base of the ladder is 1 meter from the wall, find the speed of the top of the ladder as it slides on the wall.

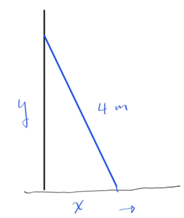
   (a) Make a sketch summarizing the information in the problem. Label the quantities you use in your solution.

   (b) Find the height of the top of the ladder when the base of the ladder is 1 meter away from the wall.

   (c) Find the speed of the top of the ladder when the base of the ladder is 1 meter from the wall. Is the top of the ladder moving up or down?

Solution:

a) We let x and y denote the distance from the base of the ladder to the wall and the height of the ladder, respectively.



b) When $x = 1$ m, we may use Pythagoras's theorem $x^2 + y^2 = 16$ and $x = 1$ to solve for $y = \sqrt{16-1}$. We choose the positive square root because y represents a length.
The top of the ladder is $\sqrt{15}$ meters above the floor.

c) Again we begin with Pythagoras's theorem, $x^2 + y^2 = 16$. We differentiate both sides to obtain $2xx' + 2yy' = 0$ and solve for y',

$$y' = -\frac{x}{y}x'.$$

Substituting $x = 1$ m, $y = \sqrt{15}$ m and $x' = 0.3$ m/s we have

$$y' = -\frac{3}{10\sqrt{15}}\,\text{m/s}.$$

The speed of the ladder is $\frac{3}{10\sqrt{15}}$ m/s and the top of the ladder is moving down the wall.

As a decimal, the speed is approximately 0.07746 m/s. The exact answer is preferred.

Grading: a) Sketch (1 point). Sketch should label at least one length.

b) Pythagoras theorem (2 points), length (1 point)

c) Differentiate Pythagoras theorem (2 points), solve for y' (1 point), value for y' (1 point).

Answer to question for speed and direction (1 point).

Units on final answers in b) and c) (1 point).

15. (10 points) Let $f(x) = xe^{-2x}$. Use calculus to find the following.
    (a) The intervals of increase or decrease.
    (b) Each value x where f has a local maximum and each value x where f has a local minimum. Explain how you determine if each is a local maximum or minimum.
    (c) The intervals of concavity.
    (d) The inflection point(s).

Solution: Compare WS 3.6.3 #4, WA 5

a) We use the product rule to compute $f'(x) = e^{-2x} - 2xe^{-2x} = (1-2x)e^{-2x}$. We have $f'(x) > 0$ if $x < 1/2$ and $f'(x) < 0$ if $x > 1/2$.

x	$(-\infty, 1/2)$	$1/2$	$(1/2, \infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$		$\searrow$

The function f is increasing $(-\infty, 1/2]$ and decreasing on the interval $[1/2, \infty)$.

b) We use the product rule again to compute $f''(x) = -2e^{-2x} - 2(1-2x)e^{-2x} = (4x-4)e^{-2x}$.

Since $f'(1/2) = 0$ and $f''(1/2) = -2/e < 0$, the function f has a local maximum at $x = 1/2$.

Alternate solution: Since $f'(x) > 0$ for $x < 1/2$ and $f'(x) < 0$ for $x > 1/2$, it follows that f has a local maximum (and global maximum) at $x = 1/2$.

c) Analyzing the sign of $f''(x) = (4x-4)e^{-2x}$, we have

x	$(-\infty, 1)$	1	$(1, \infty)$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$		$\smile$

The function f is concave down on $(-\infty, 1)$ since f'' is negative there. The function f is concave up on $(1, \infty)$ since f'' is positive there.

d) The point $(1, 1/e^2)$ is an inflection point since f changes concavity at $x = 1$.

Grading: a) $f'(x)$ (2 points), intervals of increase and decrease (1 points), use sign of f' to decide on increase or decrease (1 point).

Accept open intervals $(-\infty, 1/2)$ and $(1/2, \infty)$ as intervals for increase or decrease.

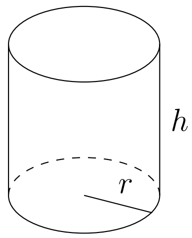
b) Compute $f''(x)$ (1 points), Local max. at $x = 1/2$ (1 point), reason (1 point).

c) Give intervals of concavity (1 points), use sign of f'' to characterize intervals (1 point)

d) List inflection point (1 point). (Do not deduct if y-coordinate omitted.)

16. (10 points) Suppose we construct a right circular cylinder radius r and height h and total surface area of 150π square centimeters. Use calculus to find the dimensions r and h of the cylinder with the largest volume.

    The volume of the cylinder is $\pi r^2 h$ and the surface area is $2\pi rh + 2\pi r^2$.



Solution: We want to maximize $V = \pi r^2 h$ with the relation $2\pi r^2 + 2\pi rh = 150\pi$.

We solve the equation for surface area for h to find $h = \dfrac{75}{r} - r$. We need $0 < r < \sqrt{75}$ for r and h to be positive. Substitute into the equation for volume to find V as $V(r) = \pi(75r - r^3)$.

Thus to find the maximum volume, we want to find the maximum value of V for r in the interval $(0, \sqrt{75})$.

We compute $V'(r) = 75 - 3r^2$ and find that the critical with $r > 0$ is $r = \sqrt{25} = 5$. For this value of r, $h = 75/5 - 5 = 10$.

This will be a maximum since $V'(r) > 0$ for $0 < r < 5$ and $V'(r) < 0$ for $5 < r < \sqrt{75}$.

The cylinder with largest volume will have $r = 5$ cm and $h = 10$ cm.

Grading: Solve area for h (2 points), substitute to express V in terms of r (2 points), find critical point (3 points), give r and h for maximum volume (2 points), units on final answer (1 point).

Problem does not ask for justification that we have found a maximum.