Name: ______________________ Section and/or TA: ______________

Do not remove this answer page. You will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or communication capabilities is permitted. You may not use a phone or other communication device during the exam.

The exam consists of 12 multiple choice questions that are worth 5 points each and 4 free response questions that are worth 10 points each. You should work the multiple choice questions on the question page. After you have checked your work carefully, record your answers by completely filling in the circle below that corresponds to your answer. If you must change your answer, make a note on the front of the exam. Be sure to check carefully when you transfer your answers to the cover sheet.

Show all work to receive full credit on the free response problems. You do not need to compute a decimal approximation to your answer. For example, the answer 4π is preferred to 12.57.

Multiple Choice Questions

1	(A)	(B)	(C)	(D)	(E)	**7**	(A)	(B)	(C)	(D)	(E)
2	(A)	(B)	(C)	(D)	(E)	**8**	(A)	(B)	(C)	(D)	(E)
3	(A)	(B)	(C)	(D)	(E)	**9**	(A)	(B)	(C)	(D)	(E)
4	(A)	(B)	(C)	(D)	(E)	**10**	(A)	(B)	(C)	(D)	(E)
5	(A)	(B)	(C)	(D)	(E)	**11**	(A)	(B)	(C)	(D)	(E)
6	(A)	(B)	(C)	(D)	(E)	**12**	(A)	(B)	(C)	(D)	(E)

SCORE

Multiple Choice	13	14	15	16	Total Score
60	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) The area of circle is increasing at a rate of 12π square units per minute. Find the rate of change (in units/minute) of the radius when the radius is 3 units.

   A. 3 **B.** 2 C. 1/2 D. 1/3 E. 1/6

   Solution: We have $A(t) = \pi r(t)^2$. Differentiating, $A' = 2\pi r r'$. Substituting $A' = 12\pi$ and $r = 3$, gives $12\pi = 2\pi \cdot 3 \cdot r'$. Solving for r' we find $r' = 2$.

   Compare WW3.2 #2.

2. (5 points) Suppose y is the solution of the initial value problem $\begin{cases} y' = 2y \\ y(0) = 3. \end{cases}$ Find the function y.

   A. $y(t) = e^{3(t-2)}$

   B. $y(t) = 3e^{2t}$

   C. $y(t) = 3e^{t-2}$

   D. $y(t) = 2e^{t-3}$

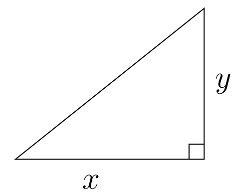
   E. $y(t) = 2e^{3t}$

   Solution: Since $y' = 2y$, we have $y(t) = Ae^{2t}$. The condition $y(0) = 3 = Ae^{2\cdot 0}$ implies $A = 3$.

   Compare WW3.3#4

3. (5 points)

In the right triangle pictured at right, the lengths of the two legs, x and y, sum to 8. What is the area of the largest possible area for a triangle that satisfies this condition?



A. 8 B. $8\sqrt{2}$ C. 16 D. 2 E. 4

Solution: The area of the triangle is $xy/2$. We are given $x+y=8$ so that $y=8-x$. Thus the area is $A(x)=x(8-x)/2$. The vertex of the graph of this quadratic function is at $(4,8)$. So that maximum area is 8 square units.

Compare WS§2.12#8

4. (5 points) The derivative of a function f is $f'(x) = x(x-1)e^x$. Select the correct statement.

A. The function f does not have a local maximum or a local minimum at 0.

B. The point $x = 0$ is a singular point for f.

C. The point $x = 0$ is not a critical point for f.

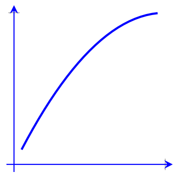
D. The function f has a local minimum at 0.

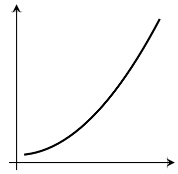
E. The function f has a local maximum at 0.

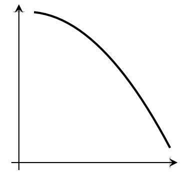
Solution: The derivative $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $0 < x < 1$. Thus f is increasing for $x < 0$ and f is decreasing for $0 < x < 1$. This implies f has a local maxmimum at $x = 0$.

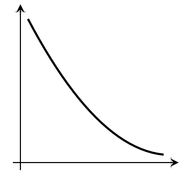
Compare WW3.6.1-2#6

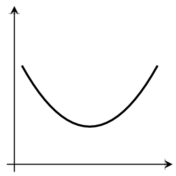
5. (5 points) Select the graph of the function which is increasing and concave down.

A. 

B. 

C. 

D. 

E. 

Solution:

Compare WS 3.6.3 #2, WW 3.6.3, #1

6. (5 points) Find the point c so that $f(c)$ is the global minimum value of $f(x) = 2x - x^2$ on the interval $[0, 4]$

A. 0 **B.** 4 C. 3 D. 2 E. 1

Solution: We know the global minimum value occurs at an endpoint or a critical point. (There are no singular points.) The endpoints are 0 and 4 and the critical point is $x = 1$. Computing, $f(0) = 0$, $f(1) = 1$ and $f(4) = -8$, we see the smallest value is -8 which occurs when $x = 4$.

Compare WW3.5.1-2#5

7. (5 points) Let $f(x) = xe^{-2x}$. Find the value c so that $(c, f(c))$ is an inflection point for f.

   A. -1 B. $1/2$ C. 0 **D.** 1 E. $-1/2$

   Solution: For $f(x) = xe^{-2x}$, $f'(x) = e^{-2x} - 2xe^{-2x} = (1-2x)e^{-2x}$ and $f''(x) = -2e^{-2x} - 2(1-2x)e^{-2x} = (4x-4)e^{-2x}$. We have that $f''(1) = 0$ and f'' changes sign at $x = 1$. Thus f has an inflection point at $(1, f(1))$.

   Compare WS §2.8, #4

8. (5 points) Suppose that the derivative of a function f is $f'(x) = x(x-1)$. Find the interval or intervals where f is decreasing.

   A. $[1, \infty)$

   B. $[0, 1]$

   C. $(-\infty, 0]$ and $[1, \infty)$

   D. $(-\infty, 0]$

   E. $(-\infty, 1]$

   Solution: We have the following information about the sign of f'.

x	$(-\infty, 0)$	0	(0,1)	1	$(1, \infty)$
$f'(x)$	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

   The function f is decreasing on the interval $(0, 1)$.

   Compare WS2.8 #3, WW2.8#2

9. (5 points) Let $f(x) = \arctan(x)$. Find the critical point(s) of f.

   A. $1, -1$

   B. 1

   C. 0

   D. There are no critical points

   E. -1

Solution: The derivative of $\arctan(x)$ is $1/(1+x^2)$. The function $1/(1+x^2) > 0$ for all x. Thus $f'(x) = 0$ has no solutions and there are no critical points.

Compare WW3.5.1-2#1.

10. (5 points) Find the value K for which the limit $\lim_{x\to\pi/2} \dfrac{\sin(x) - K}{x - \pi/2}$ exists and has a finite value.

   A. -1

   B. 1

   C. 0

   D. $-\sqrt{2}/2$

   E. $\sqrt{2}/2$

Solution: If the limit of the numerator $\lim_{x\to\pi/2}(\sin(x) - K) = \sin(\pi/2) - K$ is not zero, then the limit in the problem. If we do have $\sin(\pi/2) - K = 0$ or $K = \sin(\pi/2) = 1$, then we may use l'Hôpital's rule to find the limit as

$$\lim_{x\to\pi/2} \frac{\sin(x) - 1}{x - \pi/2} = \lim_{x\to\pi/2} \frac{\cos(x)}{1} = 1.$$

Compare WS§3.1#4.

11. (5 points) Find the limit

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{x \sin(x)}.$$

A. 2   B. 1   C. -1 **D.** 0   E. -2

Solution: Since this limit is the indeterminate form 0/0, we use L'Hôpital. One application of l'Hôpital's rule produces a second instance of the indeterminate form 0/0. A second application of l'Hôpital's rule gives the value of the limit.

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{x \sin(x)} = \lim_{x \to 0} \frac{2x \sin(x^2)}{\sin(x) + x \cos(x)} = \lim_{x \to 0} \frac{2 \sin(x^2) + 4x^2 \cos(x^2)}{\cos(x) + \cos(x) - x \sin(x)} = 0.$$

Compare WW§3.4.1-3#4

12. (5 points) The function F is an anti-derivative of x^2 and $F(1) = 1$. Find F.

A. $F(x) = 2x + 1$

B. $F(x) = \frac{x^3}{3} + \frac{2}{3}$

C. $F(x) = \frac{x^3}{3} - \frac{2}{3}$

D. $F(x) = 2x - 1$

E. $F(x) = \frac{x^3}{3} + 1$

Solution: The power rule for anti-derivatives tells us $F(x) = \frac{x^3}{3} + C$. Using the the condition $1 = F(1) = 1/3 + C$, we find $C = 2/3$ and thus $F(x) = x^3/3 + 2/3$.

Compare WW4.1 #7

Free response questions, show all work

13. (10 points) A ladder of length 5 meters is leaning against a wall. The top of the ladder is sliding down the wall at a rate of 0.2 meters/second. You should assume that the wall is perpendicular to the ground.

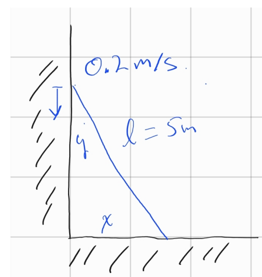
(a) Draw a sketch summarizing the information in the problem.

(b) At the time when the bottom of the ladder is 4 meters away from the wall, find the speed with which the bottom of the ladder is moving.

(c) At the time when the bottom of the ladder is 4 meters away from the wall, is the bottom of the ladder moving towards the wall or away from the wall?

Solution:

a) In the triangle at right, the sides x and y are changing and the length of the ladder ℓ is fixed.



b) The sides x, y, and ℓ are related by Pythagoras's theorem, $x^2 + y^2 = \ell^2$. Differentiating with respect to time, we find

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Solving for dx/dt.

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}.$$

When $x = 4$, we have $y = \sqrt{5^2 - 4^2} = 3$ and we are given that dy/dt is -0.2 meters/second. Thus we obtain $\frac{dx}{dt} = -\frac{3}{4}(-0.2) = \frac{3}{20}$ meters /second.

The speed is $3/20 = 0.15$ meters/second.

c) Since $dx/dt > 0$, the length x is increasing and the ladder is moving away from the wall.

Grading summary [a) 1 point, b) 8 points, c) 1 point]

a) Sketch with either length of ladder or velocity of top of ladder labelled (1 point)

b) Pythagoras's theorem relating sides (1 point), derivative of Pythagoras (2 points), solve for dx/dt (1 point), value of y (1 point), giving $dy/dt = -0.2$ m/s with correct sign (1 point). Substituting values of $x = 4$, $y = 3$ and $dy/dt = -0.2$ to find dx/dt (1 point) and give speed (1 point) as a positive value.

c) Answer (1 point). Accept correct answer even if it does not agree with the sign they found for dx/dt in part b).

Compare WW3.2 #3.

14. (10 points) Suppose that the mass of a radioactive substance X decays exponentially with a half-life of 6 years.

(a) If we have 100 grams of substance X at time $t = 0$, find A and k so that the function $m(t) = Ae^{-kt}$ gives the mass of substance X after t years. Give exact values, rather than decimal approximations for A and k.

(b) After how many years will we have 50 grams of substance X remaining? Your answer should be rounded correctly to 3 decimal places or more accurate.

(c) After how many years will we have 10 grams of substance X remaining? Your answer should be rounded correctly to 3 decimal places or more accurate.

Solution: a) We have $A = m(0) = 100$. To find k, we use that the half-life is 6 years and so $m(6) = 50 = 100e^{-k6}$. Solving for k, we have $e^{-k6} = 1/2$ or $-k6 = \ln(1/2)$ or $k == -\dfrac{\ln(1/2)}{6} = \dfrac{\ln(2)}{6}$.

b) According to the definition of half-life $m(6) = 100/2 = 50$ grams.

c) We want to find the time T so that $m(T) = 100e^{-kT} = 10$. Solving for T, we have $e^{-kT} = 1/10$ and thus $-kT = \ln(1/10)$ or $T = -\ln(1/10)/k = -6\ln(1/10)/\ln(2) = 6\ln(10)/\ln(2)$. Evaluating T as a decimal, $T \approx 19.93157$ years or 19.932 years when rounding to 3 decimal places.

Grading summary [a) 5 points, b) 2 points, c) 3 points]

a) $A = 100$, (1 point), equation for k ($e^{-k6} = 1/2$) (2 points), value for $k = \ln(2)/6$ (2 points)

b) Answer (2 points)

c) Equation for T, $e^{-kT} = 1/10$, (2 points), value for $T = 6\ln(10)/\ln(2) \approx 19.932$ or equivalent, (1 point). Accept exact or decimal answers if accurate to three or more decimal places.

Compare WSR3, WA5.

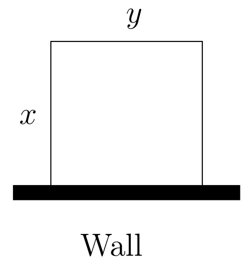
15. (10 points)

We want to build a rectangular pen. One side of the pen will be part of long, straight rock wall and the other three sides are to be constructed using 60 meters of fencing.

(a) Express the area as a function of the sidelength x in the sketch at right and give the domain of this function.

(b) Use calculus to find the dimensions and the area of the pen with the largest possible area.

(c) Explain why you have found the pen with the largest possible area.



Solution: a) The three sides made from fencing have total length 60 or $2x+y=60$. The area is $A(x)=xy=x(60-2x)$. Since x and y are non-negative, we have $x \geq 0$ and $60-2x \geq 0$ or $x \leq 30$. Thus, the area is $A(x)=60x-2x^2$ for $0 \leq x \leq 30$.

b) The largest area of $A(x)=60x-x^2$ on the interval $[0,30]$ will be at an endpoint or a critical point.

The derivative of A is $A'(x) = 60-4x$ and the critical point is the solution of $60-4x=0$ or $x=15$. To find the largest value, we consider the values at the

x	0	15	30
$A(x)$	0	450	0

The largest area is 450 square meters when $x=15$ meters and $y=30$ meters.

c) We know the largest value occurs at an endpoint, a critical point or a singular point. After checking the area at these values, the largest area is 450 square meters. Other potential explanations: –Since $A''(x)<0$ for all x, the critical is a maximum. –Since A is increasing for $x<15$ and decreasing for $x>15$, A has a maximum at $x=15$. –The graph of A is a parabola which opens down, so the vertex gives a maximum.

Grading summary. [a) 4 points, b) 5 points, c) 1 point]

a) Relation $2x+y=60$ (1 point), write area as $A(x)=60x-2x^2$ (2 points). Need $0 \leq x \leq 30$ for x and y to be positive. (1 point)

b) Find A' (1 point), critical point at $x=15$ (1 point), Values for x, y, and area (3 points). Deduct one point if no answers include units.

c) Explanation (1 point)

Compare WSR2 #5

16. (10 points) A function F satisfies $F''(x) = e^x + 6x$, $F(0) = 0$ and $F'(0) = 1$.
   (a) Find F'.
   (b) Find F.

Solution: a) Taking the anti-derivative of F'', $F'(x) = e^x + 3x^2 + C$. Using that $F'(0) = 1$, we have $e^0 + 0 + C = 1$ or $C = 0$. $F'(x) = e^x + 3x^2$.

b) Taking the anti-derivative of F', we have $F(x) = e^x + x^3 + D$. We use $F(0) = e^0 + 0 + D = 0$, we find $D = -1$. Thus $F(x) = e^x + x^3 - 1$.

Grading summary [a) 5 points, b) 5 points]

a) Anti-derivative of e^x (1 point), anti-derivative of $6x$ is $3x^2$ (2 points), value for $C = 0$ and answer to part a) (2 points).

b) Anti-derivative (1 point), anti-derivative of $3x^2$ is x^3 (2 points), value for $D = -1$ and answer to part b) (2 points).

Compare WW 4.1 #8.