Calculus I
Examination 3

10 April 2007

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please: 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*, 3) give exact answers, rather than decimal approximations to the answer.
Please write your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question. If you use the back of a sheet, indicate this by the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name Answer

Section ____________

Last four digits of student identification number ____________

Question	Score	Total
p. 1/Q1–2		14
p. 2/Q3		14
p. 3/Q4–5		14
p. 4/Q6–7		14
p. 5/Q8–9		14
p. 6/Q10		14
p. 7/Q11		14
p. 8/Q12		14
Free	2	2
		100

7 pts 1. Find the value of the following limits. Show your work. The value may be a real number, $+\infty$ or $-\infty$.

(a) $\displaystyle\lim_{x\to\infty} \frac{2x^3+10x^2}{3x^3+100/x}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x^5+x^3+2x}{-x^4-2x^2}$.

3 (a) $\displaystyle\lim_{x\to\infty} \frac{2x^3+10x^2}{3x^3+\frac{100}{x}} = \lim_{x\to\infty} \frac{2+\frac{10}{x}}{3+\frac{100}{x^4}} = \frac{2+0}{3+0} = \underline{\underline{\frac{2}{3}}}$

4 (b) $\displaystyle\lim_{x\to-\infty} \frac{x^5+x^3+2x}{-x^4-2x^2} = \lim_{x\to-\infty} \frac{1+\frac{1}{x^2}+\frac{2}{x^4}}{-\frac{1}{x}-\frac{2}{x^3}}$. Since the numerator approaches 1 whereas the denominator approaches 0 and is positive for negative x, the limit is ∞.

(a) $\frac{2}{3}$, (b) ∞

7 pts 2. Let $f(x) = x^4 - 4x^3 - 18x^2 + 5x - 7$. Find the intervals where the graph of f is concave up and the intervals where the graph of f is concave down.

1 $f'(x) = 4x^3 - 12x^2 - 36x + 5$

1 $f''(x) = 12x^2 - 24x - 36 = 12(x^2-2x-3)$

1 $\quad = 12(x-3)(x+1)$

$f''(x) = 0$ for $x=-1$ and $x=3$

4

	$-\infty < x < -1$	$-1 < x < 3$	$3 < x$
f''	$+$	$-$	$+$
f	$\cup$	$\cap$	$\cup$

Interval(s) where the graph is concave up $(-\infty,-1)$ and $(3,\infty)$

Interval(s) where the graph is concave down $(-1,3)$

(14 pts) 3. A rock is thrown upwards from a building that is 20 meters tall. One second after the ball is thrown, the rock's velocity is 5 meters/second in the upward direction. If we assume that the acceleration of gravity is 10 meters/second^2 in the downward direction, find $h(t)$, the height of the rock at all times t. Find the time when the rock hits the ground.

(1) acceleration $a(t) = -10$

(2) velocity $v(t) = -10t + C$.

(2) We know $v(1) = 5$, thus $-10 + C = 5$ and $C = 15$. Hence $v(t) = -10t + 15$.

(2) Position $h(t) = -5t^2 + 15t + D$.

(2) We know $h(0) = 20$, thus $D = 20$.

(3) The rock hits the ground when $h(t) = 0$.

$$-5t^2 + 15t + 20 = -5(t^2 - 3t - 4)$$
$$= -5(t-4)(t+1).$$

(2) The solution $t = -1$ is not relevant to this situation, thus the rock hits the ground after 4 seconds.

Of course, the position $h(t)$ of the rock is $h(t) = 0$ for $t \geq 4$.

$h(t) =$ $-5t^2 + 15t + 20$ meters for $0 \leq t \leq 4$.

Time the rock hits the ground at time $t = 4$ seconds.

7 pts

4. Find all horizontal and all vertical asymptotes of the function

$$f(x) = \frac{x^2+4}{2x^2-4x-6}.$$

Vertical asymptotes:

2 $2x^2-4x-6 = 2(x^2-2x-3) = 2(x-3)(x+1) = 0$ for $x=3$ and $x=-1$. Since the numerator is not zero for $x=3$ or $x=-1$ we have

2 $\lim_{x\to 3} f(x) = \pm\infty = \lim_{x\to -1} f(x)$.

Horizontal asymptotes:

3 $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1+\frac{4}{x^2}}{2-\frac{4}{x}-\frac{6}{x^2}} = \frac{1+0}{2-0-0} = \frac{1}{2}$.

Horizontal asymptote(s) $y = \frac{1}{2}$

Vertical asymptote(s) $x = 3$ and $x = -1$

7 pts

5. Find the function whose second derivative is $g''(x) = \sin(x) + 2\cos(x)$ and whose tangent line at $x = 0$ is given by the equation $y = 3x - 4$.

1 $g'(x) = -\cos x + 2\sin x + C$

1 $g(x) = -\sin x - 2\cos x + Cx + D$

3 The tangent line tells us: $g(0) = -4$, $g'(0) = 3$

1 Hence $3 = g'(0) = -1 + C$, thus $\boxed{C = 4}$.

1 $-4 = g(0) = -2 + D$, thus $\boxed{D = -2}$

$g(x) =$ $-\sin x - 2\cos x + 4x - 2$

8 pts

6. Circle (T) if the statement is true and (F) is false. There is no penalty for guessing and no justification is expected.

2 points each

T | F || Let $f(x) = x\sin(x)$ for x in $[23, 280]$. The function f attains an absolute maximum value.

T | F || If f is a continuous function on a closed interval $[a, b]$ and $f'(c) = 0$ at a number c in (a, b), then f has a local maximum or minimum at c.

T | F || If the second derivative $f''(x) > 0$ for x in the interval $(3, 5)$, then the first derivative f' is increasing on the interval $(3, 5)$.

T | F || If f is a polynomial and $f(0) = f(1) = 0$, then we have $f'(c) = 0$ for some c between 0 and 1.

6 pts

7. Let $f(x) = x^3 + ax^2$. For what value of a does f have a local maximum or minimum at $x = 2$? Find a and then determine if f has a local maximum or minimum at $x = 2$?

1 $f'(x) = 3x^2 + 2ax$.

2 Wanted: $f'(2) = 0$, thus $12 + 4a = 0$

1 and $\boxed{a = -3}$

2 then $f'(x) = 3x^2 - 6x$ and

$f''(x) = 6x - 6$.

Since $f''(2) > 0$, the function $f(x)$ has a local minimum at $x = 2$.

$a =$ -3

Does f have a local maximum or minimum at $x = 2$? local minimum

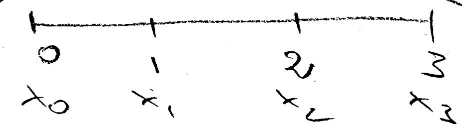
7 pts

8. Consider the integral

$$\int_0^3 (9 + 5x - 2x^2)\, dx.$$

Compute the Riemann sum for this integral which is obtained by dividing the interval $[0, 3]$ into three subintervals of equal width and using the left endpoint for each sub-interval as the sample point.

$f(x) = 9 + 5x - 2x^2$



3

$$S_3 = 1\, f(x_0) + 1 \cdot f(x_1) + 1 \cdot f(x_2)$$

2

$$= f(0) + f(1) + f(2)$$
$$= 9 + 12 + 11$$
$$= 32$$

32

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7 pts

9. Suppose that we know that

$$\int_2^5 f(x)\, dx = 5 \qquad \text{and} \qquad \int_4^5 f(x)\, dx = 2.$$

Find (a) $\int_2^4 f(x)\, dx$ (b) $\int_2^5 2f(x)\, dx$

4

$$\text{(a)} \int_2^4 f(x)\,dx = \int_2^5 f(x)\,dx - \int_4^5 f(x)\,dx = 5 - 2 = \underline{\underline{3}}$$

3

$$\text{(b)} \int_2^5 2 f(x)\,dx = 2\int_2^5 f(x)\,dx = 2 \cdot 5 = \underline{\underline{10}}$$

(a) 3 (b) 10

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

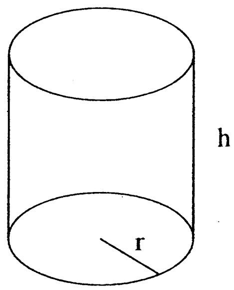
(14 pts)

10. Consider a can in the shape of a right-circular cylinder. The top and bottom of the can is made of a material that costs 4 cents per square centimeter and the side is made of a material that costs 3 cents per square centimeter. We want to find the dimensions of the can which has volume 72π cubic centimeters and whose cost is as small as possible.

    (a) Find a function $f(r)$ which gives the cost of the can in terms of the radius r. Be sure to specify the domain.

    (b) Give the radius and height of the can with least cost.

    (c) Explain how you know you have found the can of least cost.



(a) volume $= 72\pi = \pi r^2 h$, thus $\boxed{h = \frac{72}{r^2}}$.

(2)

cost function

(3)

$$\underbrace{4 \cdot 2\pi r^2}_{\text{top + bottom}} + \underbrace{3 \cdot 2\pi r \cdot h}_{\text{side}}$$

(1) thus $f(r) = 8\pi r^2 + 6\pi r \frac{72}{r^2} = 8\pi r^2 + \frac{432\pi}{r}$

(1) for $r > 0$.

(2) (b) $f'(r) = 16\pi r - \frac{432\pi}{r^2} \stackrel{!}{=} 0$. Then

(2) $16\pi r^3 = 432\pi$ and $r^3 = 27$.

thus $\boxed{r = 3\text{ cm}}$. Then $\boxed{h = 8\text{ cm}}$.

(c)

		3	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f'(r) = 16\pi\left(r - \frac{27}{r^2}\right)$

(3) $f'(r)$ is negative for all $r < 3$ and positive for all $r > 3$. This shows that $f(r)$ has an absolute minimum at $r = 3$.

14 pts

11. (a) Show how to use Newton's method to find a root of the equation $x^3 + x^2 = 7$. In your answer, you should use $x_1 = 1$ as the first approximation to the root. Give the formula relating x_{n+1} and x_n and the values of x_2 and x_3. You may round your answers to three decimal places.

(b) Below is the graph of a function f. On the graph below, illustrate two steps of Newton's method starting with the given value x_1. In your solution, you should mark the location of x_2 and x_3 on the x-axis and show how you found these values.

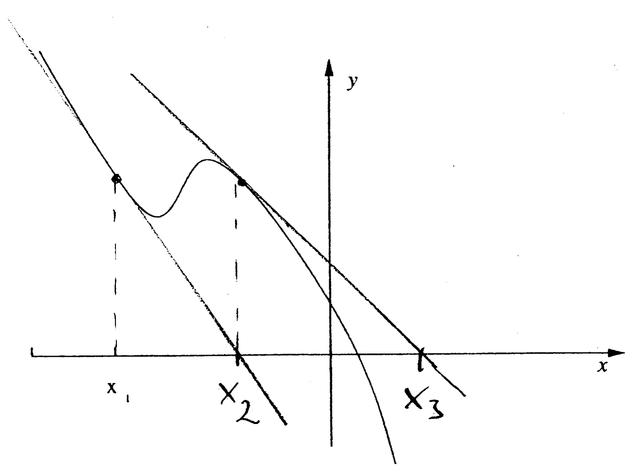
(2) (a) $f(x) = x^3 + x^2 - 7. \quad f'(x) = 3x^2 + 2x.$

(3) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n \geq 1.$$

(1) $$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-5}{5} = \underline{\underline{2}}.$$

(1) $$x_3 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{5}{16} = \underline{\underline{\frac{27}{16}}}$$

(6)



3 pts for x_2

4 pts for x_3

14 pts

12. Let $f(x) = \dfrac{4x-4}{x^2+8}$. Find the following features of the graph of f.

1 (a) y-intercept.

1 (b) x-intercept(s).

4 (c) Intervals of increase and decrease.

2 (d) Find all local maxima and minima.

2 (e) Find all horizontal asymptotes.

4 (f) On the axes below, sketch the graph of f.

1 (a) $f(0) = -\frac{4}{8} = -\frac{1}{2}$

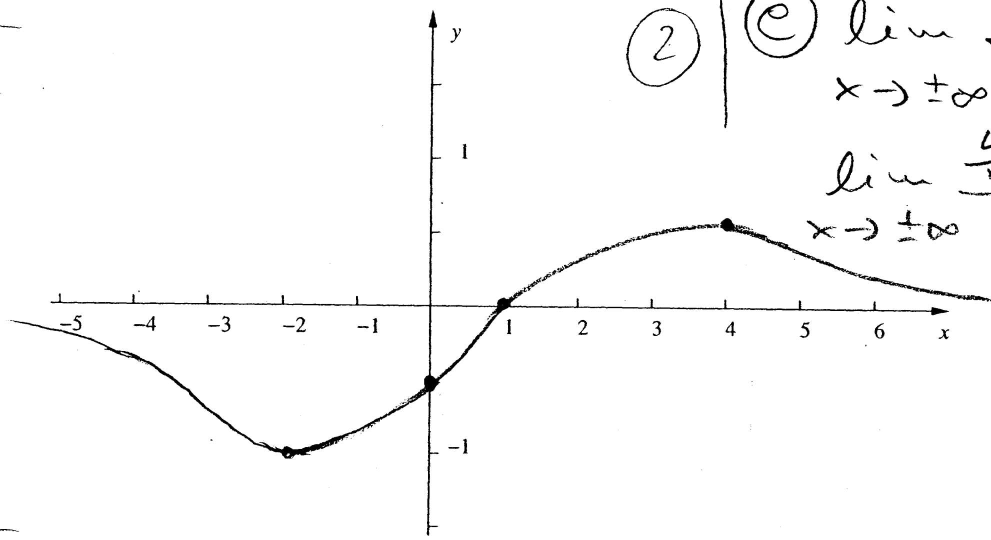
1 (b) $f(x) = 0$ for $x = 1$

2 (e) $\lim_{x \to \pm\infty} f(x) =$

$$\lim_{x \to \pm\infty} \frac{\frac{4}{x} - \frac{4}{x^2}}{1 + \frac{8}{x^2}} = 0$$

Hence $y = 0$ horizontal asymptote.

4



1 (c) $f'(x) = \dfrac{4(x^2+8) - 2x(4x-4)}{(x^2+8)^2} = \dfrac{-4x^2+8x+32}{(x^2+8)^2}$

1 critical numbers: Solve $f'(x) = 0$. Then $-4(x^2-2x-8) = 0$, thus $x = 4$ or $x = -2$

2

	-2		4	
$f'(x)$	$-$	$+$	$-$	
$f(x)$	↘	↗	↘	

f is decreasing on $(-\infty, -2)$ and $(4, \infty)$ and increasing on $(-2, 4)$.

1 (d) local min $(-2, f(-2)) = (-2, -1)$

1 local max $(4, f(4)) = (4, \frac{1}{2})$