(1) Consider the function $f(x) = 2x^3 - 3x^2 - 36x + 4$ on the interval $(-\infty, \infty)$.

(a) Find the critical number(s) of f.

(b) Find the interval(s) of increase and decrease for f.

(c) Find the local extrema of f.

a) f is always differentiable, so the critical numbers occur where $f'(x)=0$.

$f'(x) = 6x^2 - 6x - 36 \Rightarrow 6(x^2 - x - 36) = 6(x-3)(x+2) = 0$. 3

Critical numbers: $x=3$, $x=-2$

b) We do factor analysis:

Interval	$6(x-3)$	$x+2$	$f'(x) = 6(x-3)(x+2)$
$x<-2$	$-$	$-$	$+$ (increasing)
$-2<x<3$	$-$	$+$	$-$ (decreasing)
$x>3$	$+$	$+$	$+$ (increasing)

3

c) At the critical value $x=-2$, the sign of f' changes from $+$ to $-$. By the First Derivative Test f thus has a local maximum at $x=-2$, $f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 4 = 48$. 4

f' changes from $-$ to $+$ at $x=3$, so f has a local minimum at $x=3$. $f(3) = -77$.

(a) The critical number(s): $x=-2$, $x=3$

(b) Interval(s) of increase: $(-\infty, -2)$, $(3, \infty)$ and decrease: $(-2, 3)$

(c) The local maximum is at $x =$ -2 and $f(x) =$ 48

The local minimum is at $x =$ 3 and $f(x) =$ -77

(2) Find the absolute minimum value of the function

$$f(t) = t + \sqrt{1-t^2}$$

on the interval $[-1, 1]$. Be sure to also specify the value of t where the absolute minimum is achieved.

2

By the Closed Interval Test, we must compare values of f at the endpoints $t=-1$, $t=1$ and at the critical values in order to determine the absolute minimum on $[-1,1]$.

Critical values: $f'(t) = 1 + \frac{1}{2}(1-t^2)^{-1/2}(-2t) = 1 - \frac{t}{\sqrt{1-t^2}}$.

f is thus differentiable in $(-1,1)$, so we set $f'(t)=0$:

$$1 - \frac{t}{\sqrt{1-t^2}} = 0$$

$\sqrt{1-t^2} - t = 0$ and any critical number is non-negative $(t = \sqrt{1-t^2} \geq 0)$ 2

$1 - t^2 = t^2$

$1 = 2t^2 \overset{t \geq 0}{\Rightarrow} t = \frac{1}{\sqrt{2}}$ (critical number)

Comparison: Endpoints: $f(-1) = -1 + \sqrt{1-1^2} = -1$
$f(1) = 1 + \sqrt{1-1^2} = 0$. 2

Critical values: $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Absolute minimum: $f(-1) = -1$ 1

Absolute minimum -1 at $t = -1$

(3) Consider the function

$$f(x) = 2x + \sin x$$

on the interval $(-\pi, 2\pi)$.

(a) Find the interval(s) of concavity of the graph of $f(x)$; show your work.

(b) Find the point(s) of inflection of the graph of $f(x)$; justify your work.

a) The graph of f is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$.

$$f'(x) = 2 + \cos x$$

$$f''(x) = -\sin x$$

5

$f''(x) > 0$: $-\sin x > 0$ if $\sin x < 0$, that is, on $(-\pi, 0)$ and $(\pi, 2\pi)$.

$f''(x) < 0$: $-\sin x < 0$ if $\sin x > 0$, that is, on $(0, \pi)$.

b) An inflection point occurs where f'' changes sign. Thus on $(-\pi, 2\pi)$, f has inflection points at $x = 0$ (where f'' changes from $+$ to $-$) and at $x = \pi$ (where f'' changes from $-$ to $+$).

5

$f(0) = 2 \cdot 0 + \sin 0 = 0 \Rightarrow (x,y) = (0, 0)$

$f(\pi) = 2\pi + \sin \pi = 2\pi \Rightarrow (x,y) = (\pi, 2\pi)$

(a) Interval(s) where the graph is concave up: $(-\pi, 0)$, $(\pi, 2\pi)$
Interval(s) where the graph is concave down: $(0, \pi)$

(b) Point(s) of inflection (x and y coordinates): $(0,0)$ and $(\pi, 2\pi)$

(4) (a) Let $f(x) = \sqrt{16+x}$. First, find the linear approximation to $f(x)$ at $x = 0$. Then use the linear approximation to estimate $\sqrt{15.75}$. Present your solution as a rational number.

(b) Suppose that g is a differentiable function whose tangent line at $x = 2$ is given by $y = 2x - 1$. Suppose that we are given an initial approximation $x_1 = 2$ to a root c of $g(x)$. Starting with the initial approximation $x_1 = 2$, use Newton's method to find a new approximation x_2 to c.

2

a) The linearization is: $L(x) = f(a) + f'(a)(x-a)$, Here $a=0$, so

$L(x) = f(0) + f'(0)(x-0) = f(0) + f'(0)x$,

$f(0) = \sqrt{16+0} = 4$

$f'(x) = \frac{1}{2}(16+x)^{-1/2} = \frac{1}{2\sqrt{16+x}}$, so $f'(0) = \frac{1}{2\sqrt{16+0}} = \frac{1}{8}$.

Thus, $L(x) = 4 + \frac{1}{8}x$,

3

To approximate $\sqrt{15.75}$ notice that $\sqrt{15.75} = \sqrt{16 - \frac{1}{4}} = f(-\frac{1}{4})$,

But $f(-\frac{1}{4}) \approx L(-\frac{1}{4}) = 4 - \frac{1}{8}\cdot\frac{1}{4} = \frac{127}{32}$. Thus $\sqrt{15.75} \approx \frac{127}{32}$.

2

b) The basic Newton iteration is: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, so $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

To find $f(x_1) = f(2)$ and $f'(x_1) = f'(2)$, notice that the values of f and f' at $x=2$ are the same as the values of the tangent line to f at $x=2$ and its derivative. Thus $f(2) = (2x-1)|_{x=2} = 2\cdot2-1 = 3$; and $f'(2) = \frac{d}{dx}(2x-1)|_{x=2} = 2$.

5

Thus $x_2 = 2 - \frac{3}{2} = \frac{1}{2}$,

Alternatively, one Newton step starting at $x=2$ is equivalent to finding the zero of the tangent to f at $x=2$. Thus we set $2x_2 - 1 = 0$, and find $x_2 = \frac{1}{2}$.

(a) Linear approximation $L(x) =$ $4 + \frac{1}{8}x$

$\sqrt{15.75} \approx$ 127/32

(b) $x_2 =$ 1/2

(5) Find the general antiderivative of each function:

(a) $f(x) = x^2 + 1$.

(b) $g(\theta) = \dfrac{1}{\theta^2} + 2\cos\theta$.

(c) $h(t) = 2t^{1/2} + e^t$.

a) $F(x) = \frac{1}{3}x^3 + x + C$) 3

b) $g(\theta) = \theta^{-2} + 2\cos\theta$, so $G(\theta) = -\theta^{-1} + 2\sin\theta + C$) 4

c) $H(t) = 2 \cdot \frac{2}{3} \cdot t^{3/2} + e^t + C = \frac{4}{3}t^{3/2} + e^t + C$ 3

(a) $\frac{1}{3}x^3 + x + C$

(b) $-\theta^{-1} + 2\sin\theta + C$

(c) $\frac{4}{3}t^{3/2} + e^t + C$

(6) Use l'Hospital's rule to evaluate the following limits:

(a) $\lim_{x\to 0} \dfrac{\tan(\pi x)}{x}$.

(b) $\lim_{x\to\infty} e^{-x}\ln(x+3)$.

a) This is a type $\frac{0}{0}$ indeterminate form, Thus:

$$\lim_{x\to 0} \frac{\tan(\pi x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\tan \pi x}{\frac{d}{dx} x} = \lim_{x\to 0} \frac{\pi\cdot \sec^2(\pi x)}{1}$$ 3

$$= \pi \cdot \sec^2 0 = \pi$$

b) This is a type $0\cdot\infty$ indeterminant form, so we write

$e^{-x}\ln(x+3) = \dfrac{\ln(x+3)}{1/e^{-x}} = \dfrac{\ln(x+3)}{e^x}$, $\lim_{x\to\infty} \dfrac{\ln(x+3)}{e^x}$ is type $\frac{\infty}{\infty}$, so: 4

$$\lim_{x\to\infty} e^{-x}\ln(x+3) = \lim_{x\to\infty} \frac{\ln(x+3)}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x+3}}{e^x} = \lim_{x\to\infty} \frac{1}{e^x(x+3)} = 0,$$

(a) $\lim_{x\to 0} \dfrac{\tan(\pi x)}{x} =$ π

(b) $\lim_{x\to\infty} e^{-x}\ln(x+3) =$ 0

(7) Find all horizontal and vertical asymptotes of the function

$$f(x) = \frac{2x^3 + \sqrt{x^4+2}}{x^3+1}.$$

Be sure to compute all limits that are needed to support your answer.

Horizontal asymptotes: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \cdot \frac{2x^3+\sqrt{x^4+2}}{x^3+1}$

$= \lim_{x\to\infty} \frac{2+\sqrt{(x^4+2)/x^6}}{1+1/x^3} = \lim_{x\to\infty} \frac{2+\sqrt{\frac{1}{x^2}+\frac{2}{x^6}}}{1+\frac{1}{x^3}} = \frac{2}{1} = 2.$ 3

In exactly the same fashion, $\lim_{x\to-\infty} f(x) = 2$.

Thus the horizontal asymptote is $y=2$. 2

Vertical asymptote: f is defined except at $x=-1$, so we check to see whether $x=-1$ is a vertical asymptote. The numerator $2x^3+\sqrt{x^4+2}$ is negative when x is at or near -1, If $x<-1$, then $x^3+1<0$ also, so $\lim_{x\to-1^-} f(x) = +\infty$. 1 If $x>-1$, then $x^3+1>0$, and $\lim_{x\to-1^+} f(x) = -\infty$. 1

Thus $x=-1$ is a vertical asymptote 2

Equation(s) of horizontal asymptote(s): $y=2$

Equation(s) of vertical asymptote(s): $x=-1$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) State the Mean Value Theorem. Use complete sentences.

(b) Assume that f and g are differentiable functions on $(-\infty, \infty)$, that $f'(x) = g'(x)$ for all x, that $f(x) = \sin x$, and that $g(0) = 1$. What is $g(x)$?

(c) Suppose that g is differentiable for all x and that $-5 \le g'(x) \le 2$ for all x. Assume also that $g(0) = 2$. Based on this information, is it possible that $g(2) = 8$? Use the Mean Value Theorem to justify your answer.

a) Let f satisfy;

1) f is continuous on a <u>closed</u> interval $[a,b]$, and

2) f is differentiable on the <u>open</u> interval (a,b), 3

Then there is a number c in (a,b) such that 1

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$ 2

or alternatively,

$$f(b)-f(a) = f'(c)(b-a)$$

b) The Mean Value Theorem has as a consequence that if $f'(x) = g'(x)$ for all x, then f and g differ by at most a constant. Thus 2

2 $(g(x) = \sin x + C$ for some constant C. But $1 = g(0) = \sin 0 + C = C$.

Thus $g(x) = \sin x + 1$ 1

c) If $g(0) = 2$ and $g(2) = 8$, then by the Mean Value Theorem there must be some $c \in (0,2)$ such that $g'(c) = \dfrac{g(2)-g(0)}{2-0} = \dfrac{8-2}{2-0} = \dfrac{6}{2} = 3$, 3

But $g'(x) \le 2$ for all x, and in particular, for all c between 0 and 2

Thus it is <u>not</u> possible that $g(2) = 8$. 2

(b) $g(x) =$ $\sin x + 1$

(c) yes / no (*circle the correct answer*)

(9) In parts (b) and (c), be sure to fully justify all steps you take to compute the given limits.

(a) State l'Hospital's Rule. Use complete sentences.

(b) Find $\lim_{x\to 1} \frac{x-1}{\ln(x+1)}$.

(c) Find $\lim_{x\to 0+} x^x$.

1 2

a) Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). Suppose that either:

1) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, or

2) $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, 2

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, if this limit exists or is $\pm\infty$. 2

b). This limit is NOT an indeterminate form. By continuity, 3

$$\lim_{x\to 1} \frac{x-1}{\ln(x+1)} = \frac{1-1}{\ln(1+1)} = \frac{0}{\ln 2} = 0.$$ 2

c) $\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} \ln x^x}$. But $\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \ln x$,

2 which is a $0\cdot\infty$ indeterminate form. Thus $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$ 1

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x\to 0^+} -x = 0$. Then $\lim_{x\to 0^+} x^x = e^0 = 1$.

l'Hospital

1 1

(b) $\lim_{x\to 1} \frac{x-1}{\ln(x+1)} =$ 0 (c) $\lim_{x\to 0+} x^x =$ 1

(10) Find the point(s) on the hyperbola $y = \frac{16}{x}$ that is (are) closest to the point $(0,0)$. Be sure to state clearly what function you choose to minimize or maximize and why.

6 The distance from a point (x,y) on the hyperbola to $(0,0)$ is given by $dist(x,y) = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$. But $y = \frac{16}{x}$, so $dist(x,y) = \sqrt{x^2 + \frac{16^2}{x^2}} = d(x)$. Note that $d(x)$ will take on its minimum at the same point ~~s~~ x where $f(x) = d(x)^2 = x^2 + \frac{16^2}{x^2}$ takes on it minimum, so we will minimize $f(x) = x^2 + \frac{16^2}{x^2}$.

3 To find critical numbers, we find $f'(x) = 2x + 16^2 \cdot x^{-3} \cdot (-2) = 2x - 2 \cdot 16^2 \cdot x^{-3}$. $f'(x) = 0 = 2x - 2\cdot 16^2 \cdot x^{-3}$ when $16^2 = x^4$, or $x = \pm 4$.

4 When $x > 0$ and $x < 4$, $f'(x) = 2x - 2\cdot 16^2/x^3 < 0$, Also, when $x > 4$, $f'(x) > 0$. Thus on $(0,\infty)$ f has a global maximum at $x = 4$ by the First Derivative Test for Global Extreme Values.
On $(-\infty, 0)$: $f'(x) < 0$ when $x < -4$, and $f'(x) > 0$ when $x > -4$, Thus f has a global minimum on $(-\infty, 0)$ at $x = -4$, Since $f(4) = f(-4)$,

1 (f is minimized on its whole domain $(-\infty, 0) \cup (0,\infty)$ at $x = 4$ and $x = -4$.

2 (Thus the points on $y = \frac{16}{x}$ closest to $(0,0)$ are $(4,4)$ and $(-4,-4)$

Point(s) $(x, y) =$ $(4,4)$ and $(-4,-4)$.

Extra Credit Problem.

Mark the correct answers below. For each correct answer you earn 2 points, and for each incorrect answer 1 point will be subtracted. Therefore, **it might be wise to skip a question rather than risking losing a point.** However, your final score on this problem will not be negative! You need not justify your answer.

True	False	Statement
☐	☒	If $f'(c) = 0$ and f' changes from negative to positive at c, then f must have an absolute minimum at c.
☒	☐	There exists a function $f(x)$ such that $f(x) < 0$, $f'(x) > 0$, and $f''(x) < 0$ for all x.
☐	☒	Suppose that a differentiable function f is increasing, and $f(x) > 0$ for all x. Then $g(x) = \dfrac{1}{f(x)}$ is also an increasing function.
☐	☒	If f is odd, then f' is also odd.
☒	☐	If the function f is differentiable, $f(1) = 1$, and $f(3) = -3$, then there exists a number c between 1 and 3 such that $f'(c) = -2$.