MA 113 — Calculus I Spring 2010
Exam 3 April 13, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which of problem 8 - 10 is not to be graded by crossing through its number in the table below. Answer as many extra credit problems as you wish to; please carefully read the instructions on the last page of the exam.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Answer Key

Section: ____

Last four digits of student identification number: ____

Question	Score	Total
1		10
2		8
3		8
4		12
5		12
6		10
7		10
8		15
9		15
10		15
Extra Credit		10
		100

(1) Consider the function $f(x) = \frac{x^4}{4} - 3x^3 + 10x^2 + 23$ on the interval $(-\infty, \infty)$.

(a) Find the critical numbers of $f(x)$.

(b) Find the interval(s) of increase and decrease for $f(x)$.

(c) Find the local extrema for $f(x)$; for each extremum give both coordinates.

(a) $f'(x) = x^3 - 9x^2 + 20x$ (1 pts)

$f'(x) = 0 \Rightarrow x(x^2-9x+20) = x(x-5)(x-4) = 0$ (1 pt)

So, critical numbers are $0, 4, 5$, (1 pt)
since Domain$(f) = \mathbb{R}$. (1 pt)

(b) Need to test sign of f' on $(-\infty, 0)$, $(0,4)$, $(4,5)$ and $(5,\infty)$. (1 p

$(-\infty, 0)$ — $f' < 0 \Rightarrow$ decreasing (1 pt for dec)

$(0,4)$ — $f' > 0 \Rightarrow$ Increasing

$(4,5)$ — $f' < 0 \Rightarrow$ decreasing (1 pt for inc)

$(5,\infty)$ — $f' > 0 \Rightarrow$ Increasing

(c) $x=0$, f' changes dec to inc, so local min at $(0,23)$ (1 pt)
$x=4$, f' changes inc to dec, so local max at $(4, 55)$ (1 pt)
$x=5$, f' changes dec to inc, so local min at $(5, \frac{217}{4})$ (1 pt)

(a) Critical numbers: 0, 4, 5

(b) Interval(s) of increase and decrease: increase $(0,4)$, $(5,\infty)$, decrease $(-\infty,0)$, $(4,5)$

(c) Local maxima (both coordinates): $(4,55)$

Local minima (both coordinates): $(0,23)$, $(5, \frac{217}{4})$

(2) Find the linear approximation, $L(x)$, to $f(x) = \sqrt{1-2x}$ at $x = -4$.

We know $L(x) = f(-4) + f'(-4)(x-(-4))$

$= f(-4) + f'(-4)(x+4)$ 2pts

$f(-4) = \sqrt{1-2\cdot(-4)} = \sqrt{9} = 3$ 2 pts

$f'(x) = \dfrac{1}{2\sqrt{1-2x}} \cdot -2 = \dfrac{-1}{\sqrt{1-2x}}$ 1pt

So, $f'(-4) = \dfrac{-1}{\sqrt{1-2(-4)}} = \dfrac{-1}{\sqrt{9}} = \dfrac{-1}{3}$ 1pt

Thus, $L(x) = 3 - \frac{1}{3}(x+4)$ 2pts

$= \dfrac{-1}{3}x + \dfrac{5}{3}$ acceptable either way

Linear approximation $L(x) =$ $-\frac{1}{3}x + \frac{5}{3}$

(3) Evaluate $\lim_{x\to\infty} \frac{3x-1}{\sqrt{5x^2-1}}$. Show all your work.

First, divide num & den by $\frac{1}{x}$. 2pts

$$\lim_{x\to\infty} \frac{3x-1}{\sqrt{5x^2-1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3-\frac{1}{x}}{\sqrt{\frac{1}{x^2}} \cdot \sqrt{5x^2-1}}$$

allowed since $x > 0$ 1pt

$$= \lim_{x\to\infty} \frac{3-\frac{1}{x}}{\sqrt{5-\frac{1}{x^2}}} = \frac{3-\lim_{x\to\infty}\frac{1}{x}}{\sqrt{5-\lim_{x\to\infty}\frac{1}{x^2}}} = \frac{3-0}{\sqrt{5-0}} = \frac{3}{\sqrt{5}}$$

2pts for getting to this point

1pt for moving limits.

2pts

$\lim_{x\to\infty} \frac{3x-1}{\sqrt{5x^2-1}} =$ $\frac{3}{\sqrt{5}}$

(4) Find the most general anti-derivative of the following functions:

(a) $f(x) = 2x + x^2$

(b) $g(x) = \frac{5}{x} + 3\sin(x)$

(c) $h(x) = 2x^2 - \sec^2(x)$

(a) $F(x) = \underbrace{x^2 + \frac{x^3}{3}}_{2\text{pts}} + \underbrace{C}_{2\text{pts}}$

(b) $G(x) = 5\ln|x| - 3\cos(x) + C$

1pt 1pt 1pt 1pt

(c) $H(x) = \underbrace{\frac{2x^3}{3}}_{1\text{pt}} - \underbrace{\tan(x)}_{2\text{pts}} + \underbrace{C}_{1\text{pt}}$

(a) General anti-derivative for f: $x^2 + \frac{x^3}{3} + C$

(b) General anti-derivative for g: $5\ln|x| - 3\cos x + C$

(c) General anti-derivative for h: $2\frac{x^3}{3} - \tan x + C$

(5) Find all horizontal and vertical asymptotes of the function $f(x) = \frac{3x^3+1}{x^3-1}$. Show all your work.

Check $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, and where denominator $= 0$.

① $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^3+1}{x^3-1} = \lim_{x\to\infty} \frac{3+\frac{1}{x^3}}{1-\frac{1}{x^3}} = 3.$

$\boxed{\text{For both:}}$ 1pt for checking, 1pt for dividing by $\frac{1}{x^3}$, 2pts for 3.

② $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x^3+1}{x^3-1} = \lim_{x\to-\infty} \frac{3+\frac{1}{x^3}}{1-\frac{1}{x^3}} = 3$

③ 1pt for checking <u>either of</u> $\lim_{x\to 1^-} f(x)$ or $\lim_{x\to 1^+} f(x)$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{3x^3+1}{x^3-1} = -\infty$ 3pts

$\boxed{\text{OR}}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{3x^3+1}{x^3-1} = \infty$ 3pts

Horizontal asymptote(s): $y = 3$

Vertical asymptote(s): $x = 1$

(6) Find the coordinates of all points on the graph of $f(x) = x^2 - 4x + 7$ where the absolute maximum and absolute minimum values occur in the interval $[1, 4]$.

Calculate $f'(x) = 2x - 4$. (1pts)

Set $f'(x) = 0 \Rightarrow x = 2$ (2pts).
to find critical value.

Evaluate
$$\begin{cases} f(1) = 4 & \text{(1pt)} \\ f(2) = 3 & \text{(1pt)} \\ f(4) = 7 & \text{(1pt)} \end{cases}$$

Abs. max is $(4, 7)$ (2pts)

Abs. min is $(2, 3)$ (2pts)

Coordinates of all points where an absolute maximum occurs: $(4, 7)$

Coordinates of all points where an absolute minimum occurs: $(2, 3)$

(7) Consider the function $f(x) = 2\cos(x) + 5$ on the interval $[0, 2\pi]$. Find numbers a and b such that $a < b$ and $f(x)$ does not change concavity on each of the intervals $(0, a)$, (a, b) and $(b, 2\pi)$.
For each of the three intervals determine whether $f(x)$ is concave up or concave down.

$f'(x) = -2\sin(x)$, $f''(x) = -2\cos(x)$. 2pts

Need to check where $f''(x) = 0$ on $[0, 2\pi]$:

$-2\cos(x) = 0$ at $x = \frac{\pi}{2}, \frac{3\pi}{2}$ 2pts 2pts

NOTE: $f''(x) < 0$ on $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$.

$f''(x) > 0$ on $(\frac{\pi}{2}, \frac{3\pi}{2})$.

So, f does not change concavity on these intervals

2pts for checking.

f is conc. up on $(\frac{\pi}{2}, \frac{3\pi}{2})$. 1pt

f is conc. down on $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$. 1pt.

$a = \frac{\pi}{2}$ $\qquad$ $b = \frac{3\pi}{2}$

f is concave up on $(\frac{\pi}{2}, \frac{3\pi}{2})$

f is concave down on $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) State the Mean Value Theorem. Use complete sentences.

Suppose f is cts on $[a,b]$ 2pts

and diff on (a,b). 2pts

Then there is a c in (a,b) 1pt

such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. 2pts

(b) Suppose that $f(x)$ is a function differentiable on $(-\infty, \infty)$ such that $f(0) = -2$ and $f'(x) \leq 6$ for all values of x. How large can $f(2)$ possibly be?

1pt Applying mean value theorem, there is some c in $(0,2)$ s.t.

3pts $f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{f(2)-(-2)}{2}$

$\Rightarrow f(2) = 2f'(c) - 2$.

4pts So, $f(2) \leq 2 \cdot 6 - 2 = 10$.

$f(2)$ cannot be larger than: 10

(9) (a) State L'Hospital's Rule for limits in indeterminate form of type $\frac{0}{0}$. Use complete sentences and include all necessary assumptions.

Suppose f + g are diff, (1pt)

and $g'(x) \neq 0$ (1pt)

on an open interval I containing a. (1pt)

If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ (2pt)

(1pt)

if right hand limit exists (1pt)
(or is $\pm\infty$).

(b) Evaluate $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$.

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} \underset{L'H}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} \underset{L'H}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

(1pt) (1pt) (2pts)

(c) Evaluate $\lim_{x\to 0^+} x^3 \ln(x)$.

$\lim_{x\to 0^+} x^3 = 0$, $\lim_{x\to 0^+} \ln(x) = -\infty$. So, change to $\lim_{x\to 0^+} \frac{\ln(x)}{x^{-3}}$ (2pt)

Now, L'H $\Rightarrow$ $$\lim_{x\to 0^+} \frac{\ln(x)}{x^{-3}} \underset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{-3x^{-4}} = \lim_{x\to 0^+} -\frac{1}{3}x^3 = 0.$$

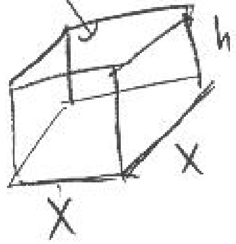
(1pt) (1pt)

(b) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2} =$ $\frac{1}{2}$

(c) $\lim_{x\to 0^+} x^3 \ln(x) =$ 0

No top!

(10) A box is to have a square base, no top, and a volume of 10 cubic centimeters. What are the dimensions of the box with the smallest possible total surface area? Provide an exact answer; do not convert your answer to decimal form.
Make a sketch and introduce all the notation you are using.



2pts

Base length : x 1pt

Height : h 1pt

Surface Area : A 1pt

Algebraic relations: $A(x,h) = x^2 + 4xh$. 1pts

$V(x,h) = x^2 h$. 1pts

Since $V = 10$, we have $h = \frac{10}{x^2}$. 2pts

1pt

Substitute: $A(x) = x^2 + \frac{40}{x}$.

1pt $A'(x) = 2x - \frac{40}{x^2}$. $A'(x) = 0 \Rightarrow 20 = x^3$ 1pt

$\Rightarrow x = \sqrt[3]{20}$.

$A'(x) > 0$ for $x > \sqrt[3]{20}$, $A'(x) < 0$ for $x < \sqrt[3]{20}$. 1pt

So, $x = \sqrt[3]{20}$ is an abs. min. 1pt

Thus, dimensions are $x = \sqrt[3]{20}$, $h = \frac{10}{(\sqrt[3]{20})^2}$ 1pt

Dimensions of box with smallest surface area: $\sqrt[3]{20} \times \frac{10}{\sqrt[3]{20}^2}$ cm^2

EXTRA CREDIT PROBLEMS: Circle the correct answers below. For each correct answer you earn 2 points, and for each incorrect answer 1 point will be subtracted. Therefore, it might be wise to skip a question rather than risking losing a point. However, your final score on this problem will not be negative! You need not justify your answer.

True or **False** : If $f'(c) = 0$ and f' changes from positive to negative at c, then f has an inflection point at c.

True or **False** : If $f'(c) = 0$ and f' changes from negative to positive at c, then f has a local minimum at c.

True or **False** : If f is differentiable on $(-\infty, \infty)$ satisfying $f(0) = 2$ and $f(3) = 4$, then there must be a point c in $(0, 3)$ satisfying $f'(c) = \frac{2}{3}$.

True or **False** : If $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, then $f(x)$ has a vertical asymptote at each point c where $q(c) = 0$.

True or **False** : If $f(x)$ is differentiable and concave up on $(-\infty, \infty)$, then the graph of $f(x)$ lies above every tangent line to $f(x)$.