MA 113 Calculus I Spring 2014
Exam 3 Tuesday, 15 April 2014

Name: Key

Section: ______________________

Last 4 digits of student ID #: ________

This exam has ten multiple choice questions (five points each) and five free response questions (ten points each). Additional blank sheets are available if necessary for scratch work. No books or notes may be used. Turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities.

On the multiple choice problems:

- Select your answer by placing an X in the appropriate square of the multiple choice answer box on the front page of the exam.
- Carefully check your answers. No credit will be given for answers other than those indicated on the *multiple choice answer box*.

On the free response problems:

- Clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
- Give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each free response question is followed by space to write your answer. Please write your solutions neatly in the space below the question.

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

Exam Scores

Question	Score	Total
MC		50
11		10
12		10
13		10
14		10
15		10
Total		100

Record the correct answer to the following problems on the front page of this exam.

1. Let $f(x) = x^3 - 2x^2$. Find the largest open interval on which f is decreasing.

   (A) $(0, 4/3)$
   (B) $(0, 2)$
   (C) $(-\infty, 2/3)$
   (D) $(2/3, \infty)$
   (E) $(-2, 0)$

f decr $\Leftrightarrow$ $f' < 0$.

$f'(x) = 3x^2 - 4x = x(3x-4)$

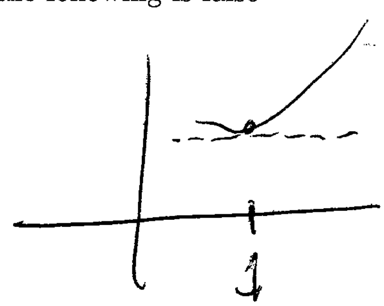
2 pts. $f'(c) = 0$ $\quad x = 0$ & $x = 4/3$.

	$x<0$	$0 < x < 4/3$	$x > 4/3$
sign(f')	pos	neg	pos

so $\quad x \in (0, 4/3)$ $\quad$ (A)

2. If $f'(1) = 0$ and $f''(1) > 0$, then which of the following is false

   (A) f has a critical point at 1.
   (B) f has a local minimum at 1
   (C) f has a local maximum at 1
   (D) f is differentiable at 1
   (E) f is continuous at 1



$f''(1) > 0$

$f'(1) = 0$

(C) since $f'' > 0$ means f is concave up near 1 so a local min.

Record the correct answer to the following problems on the front page of this exam.

3. Give the linear approximation to $f(x) = \sqrt{3x+3}$ at $x = 2$.

(A) $L(x) = \frac{1}{2}x + 2$

(B) $L(x) = \frac{1}{2}x$

(C) $L(x) = \frac{1}{2}x + \frac{1}{2}$

(D) $L(x) = \frac{3}{2}x$

(E) $L(x) = \frac{3}{2}x - \frac{5}{2}$

$f'(x) = \frac{1}{2}(3x+3)^{-\frac{1}{2}}\,3 = \frac{3}{2}(3x+3)^{-\frac{1}{2}}$

$f'(2) = \frac{1}{2}$

$f(2) = 3$

$\frac{y - f(2)}{x-2} = \frac{1}{2}$ or $y(x) = \frac{1}{2}(x-2) + f(2)$

$= \frac{1}{2}x - 1 + 3 = \frac{1}{2}x + 2$. (A)

4. Let $f(x) = \sin(2x)$. On which subintervals of $[0, \pi]$ is f concave down?

(A) $(0, \pi/4), (3\pi/4, \pi)$

(B) $(\pi/4, 3\pi/4)$

(C) $(0, \pi/2)$

(D) $(\pi/2, \pi)$

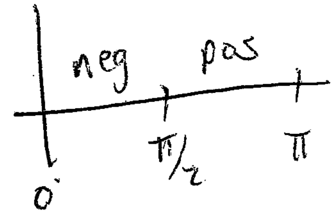
(E) $(0, \pi)$

$f'(x) = 2\cos(2x)$

$f''(x) = -4\sin(2x)$

$f''(c) = 0$ for $\in [0, \pi]$ if $x = \pi/2$

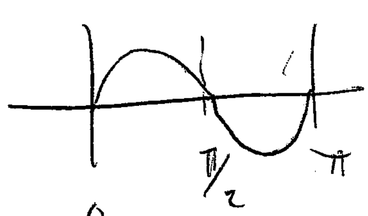
f''



So $f'' < 0$ for concave down

$\Rightarrow (0, \pi/2)$ (C)

You can also sketch $\sin(2x)$



down up -

Record the correct answer to the following problems on the front page of this exam.

5. Let $f(x) = x^3 + 3x$. Which of the following statements is true?

   (A) f has no local extrema

   (B) f has a local maximum at -1

   (C) f has a local maximum at 1

   (D) f has a local minimum at 1

   (E) f has a local minimum at -1

$f'(x) = 3x^2 + 3 = 3(x^2+1)$ so no critical pts.

(f is a polynomial)

whence no local extrema (A)

6. Let $f(x) = \dfrac{(2x+1)(x-2)(1-x)}{x^3}$. and find $\lim_{x\to\infty} f(x)$.

   (A) 0

   (B) 2

   (C) -2

   (D) ∞

   (E) $-\infty$

$f(x) = x^3(2+\frac{1}{x})(1-2/x)(\frac{1}{x}-1)/x^3$

$= -(2+\frac{1}{x})(1-2/x)(1-\frac{1}{x})$

Now take $x \to +\infty$ to get -2

(C)

Record the correct answer to the following problems on the front page of this exam.

7. If $\sum_{k=1}^{n} a_k = n^2 + n$, find $\sum_{k=11}^{20} a_k$.

(A) 42

(B) 110

(C) 132

(D) 310

(E) 420

$$\sum_{k=11}^{20} a_k = \sum_{k=1}^{20} a_k + \sum_{k=1}^{10} a_k$$

$$= (20^2+20) - (10^2+10)$$

$$= 420 - 110$$

$$= 310 \quad D$$

8. Let $f(x) = x^3 - 2$. Use Newton's method to find a solution of $f(x) = 0$ beginning with $x_0 = 2$. Give a decimal approximation of x_2, correctly rounded to three decimal places.

(A) -2.444

(B) 1.208

(C) 1.260

(D) 1.296

(E) 1.500

$$f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left[\frac{(x_n^3 - 2)}{3x_n^2}\right]$$

Record the correct answer to the following problems on the front page of this exam.

9. Let $f(x) = x^3$ on the interval $[-1, 1) = \{x : -1 \le x < 1\}$. Find the absolute maximum and minimum values of f on the interval $[-1, 1)$.

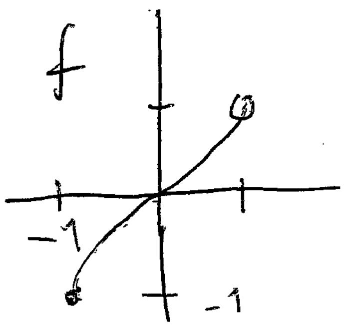
   (A) The absolute minimum value is 1 and the absolute maximum value is -1.

   (B) The absolute minimum value is -1 and the absolute maximum value is 1.

   (C) The absolute minimum value is -1 and the absolute maximum value is 0.

   (D) The absolute minimum value is -1 and there is no absolute maximum value.

   (E) There is no absolute minimum value and the absolute maximum value is 0.



Abs min at $x = -1$, $f(-1) = -1$

No Abs max since $x = 1$ is not in the domain

$f'(x) = 3x^2$ vanishes at $x = 0$

but this isn't an extrema (D)

10. Let $f(x) = x^2$. Divide the interval [0,2] into three subintervals of equal length and compute R_3, the 3rd right-endpoint approximation to the area of the region $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le x^2\}$.

   (A) 40/27

   (B) 56/27

   (C) 112/27

   (D) 22/9

   (E) 8/3

$N = 3$

$a = 0$ 1 $2 = b$

$x_1 = 2/3$ $4/3 = x_2$ $6/3$

$\Delta X = \frac{b-a}{3} = \frac{2-0}{3} = \frac{2}{3}$

$R_3 = \frac{2}{3}\left[f(2/3) + f(4/3) + f(2) \right]$

$= \frac{2}{3}\left[\frac{4}{9} + \frac{16}{9} + \frac{36}{9} \right]$

$= \frac{2}{3} \cdot \frac{56}{9} = \frac{112}{27}$ (C)

11. (a) State the mean value theorem.

(b) For each function and interval determine if the mean value theorem applies. If the theorem does apply, state this. If the theorem does not apply, explain which hypothesis fails.

i. $f(x) = x\sin(x)$ on the interval $[2, 42]$.

ii. $g(x) = |x|$ on the interval $[-1, 1]$.

a) MVT let $f: [a,b] \to \mathbb{R}$ be continuous and differentiable on $[a,b)$. Then there exists at least one point c with $a < c < b$ where $f'(c) = \frac{f(b) - f(a)}{b-a}$.

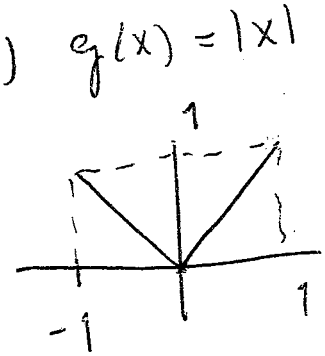
5 pts

b) (i) $f(x) = x \sin x$ on $[2, 42]$. f is continuous everywhere and differentiable everywhere so the hypotheses on f are satisfied. Also the interval $[2, 42]$ is closed and finite. So there is a $2 < c < 42$ where

$$f'(c) = \frac{f(42) - f(2)}{40} = \frac{42\sin 42 - 2\sin 2}{40}.$$

2 pts

(ii) $g(x) = |x|$ on $[-1, 1]$



g is continuous on $[-1, 1]$ but not differentiable at $x = 0$. So the hypothesis of the MVT on g is not satisfied & the MVT does not apply.

3 pts

Free Response Questions: Show your work!

12. Let $f(x) = e^{2x}$.

   (a) Find $L(x)$, the linearization of $f(x)$ at 0. Put your answer in the form $L(x) = mx + b$.

   (b) Find

$$\lim_{x \to 0} \frac{f(x) - L(x)}{x^2}.$$

(a) $f(x) = e^{2x}$, $f(0) = 1$

2 pts $f'(x) = 2e^{2x}$, $f'(0) = 2$

2 pts $L(x)$: $\dfrac{\cancel{y}L(x) - f(0)}{x - 0} = f'(0) = 2$

$L(x) - 1 = 2x \Rightarrow \boxed{L(x) = 2x + 1}$

1 pt.

(b) At $x = 0$, $f(x) - L(x) = 1 - 1 = 0$ so $\dfrac{f(x) - L(x)}{x^2}$

1 pt is "$\frac{0}{0}$" type indeterminate form at $x = 0$.

2 pts Apply l'Hopital's Rule $\lim\limits_{x \to 0} \dfrac{f(x) - L(x)}{x^2} = \lim\limits_{x \to 0} \dfrac{f'(x) - L'(x)}{2x}$

$$= \lim_{x \to 0} \frac{2e^{2x} - 2}{2x} = \lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

Look at $\dfrac{e^{2x} - 1}{x}$ at $x = 0$. It is still "$\frac{0}{0}$" type.

2 pts Apply l'Hopital's Rule again!

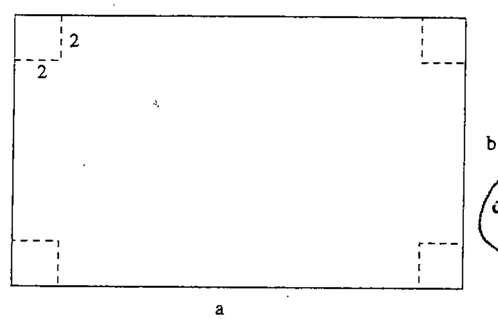
$$\lim_{x \to 0} \frac{f(x) - L(x)}{x^2} = \lim_{x \to 0} \frac{2e^{2x}}{1} = \lim_{x \to 0} 2e^{2x} = \underline{\underline{2}}.$$

Free Response Questions: Show your work!

13. We have a rectangular piece of cardboard of area 200 cm^3. A box with no top is to be constructed by removing squares of side length 2 cm from each corner and folding up the remaining flaps.

    (a) If the rectangle is a cm $\times$ b cm, find a function $V(a)$ which gives the volume of the box as a function of a. For which values of a is it possible to construct a box?

    (b) Find the dimensions a and b which give the box of largest volume and explain how you know you have found the largest volume.



a) Subtracting 4 from a & b,

2 pts $V(a,b) = 2 \times (b-4) \times (a-4)$

clearly both $a, b \geq 4$.

Constraint:

1 pt. Area $= 200\text{ cm}^2 = ab$

so $\boxed{b = 200/a \text{ cm}}$

Substitute:

1 pt. $V(a) = 2\left(\frac{200}{a} - 4\right)(a-4)$

$a \geq 4$ and $b \geq 4 \Rightarrow$ $a \leq 50$ from $b = 200/a$.

1 pt. $a \in [4, 50]$

b.) $V(a) = 2\left(200 - 4a - \frac{800}{a} + 16\right) = 432 - 8a - \frac{1600}{a}$ 1pt

2pts. $V'(a) = -8 + \frac{1600}{a^2} = 0$ so $a^2 = 200 \Rightarrow a = +\sqrt{200} = 10\sqrt{2} > 4$.

2 ways to verify absolute max occurs at $a = 10\sqrt{2}$.

(1) Check $V(a)$ at endpts. $V(4) = 0 = V(50)$ and $V(10\sqrt{2}) = 2(10\sqrt{2}-4$ > 0

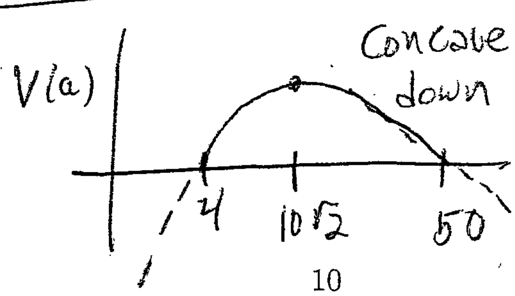
so this is the global max: $\boxed{a = b = 10\sqrt{2} \text{ cm}}$

2pts or

(2) Note $V(a)$ is defined for $a > 0$

$V''(a) = -\frac{3200}{a^3} < 0 \Rightarrow$ concave down

so a global max at $10\sqrt{2}$.



Free Response Questions: Show your work!

14. Suppose that a particle moves so that at time t seconds, its acceleration is $a(t) = 6t - 2$ cm/second^2. The position at time $t = 0$ is 7 cm to the right of the origin and the velocity at time $t = 1$ is 2 cm/second.

    (a) Find a function which gives the position at all times t.

    (b) Find the velocity at $t = 2$.

(a) Use anti-derivatives: $v(t) = \int a(t)\,dt$ since $v'(t) = a(t)$

2 pts $v(t) = \int (6t-2)\,dt = 6\int t\,dt - 2\int dt = 3t^2 - 2t + C$ (cm/sec)

Check: $v'(t) = 6t - 2$ ✓

Condition: $v(1) = 2$ cm/sec $= 3 - 2 + C = 1 + C \Rightarrow C = 1$

2 pts $\boxed{v(t) = 3t^2 - 2t + 1 \text{ cm/sec.}}$

$s(t) = \int v(t)\,dt$ since $s'(t) = v(t)$.

2 pts $= \int (3t^2 - 2t + 1)\,dt = t^3 - t^2 + t + D$

Condition $s(0) = 7$ cm $= D$.

2 pts $\boxed{s(t) = t^3 - t^2 + t + 7 \text{ cm.}}$ Check: $s'(t) = 3t^2 - 2t +$
$s''(t) = 6t - 2$ ✓

(b) $v(t=2) = 3 \cdot 4 - 2 \cdot 2 + 1 = \boxed{9 \text{ cm/sec.}}$

2 pts

15. You may find one or more of the following formulæ useful for this problem.

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \qquad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}.$$

Consider the sum

$$S_N = \sum_{k=1}^{N} (2 + \frac{3k}{N}).$$

(a) Find a closed from expression for S_N.

(b) Find the limit $\lim_{N\to\infty} \frac{1}{N} S_N$.

(a) $S_N \overset{2\text{ pts}}{=} 2\left(\sum_{k=1}^{N} 1\right) + \frac{3}{N}\left(\sum_{k=1}^{N} k\right) \overset{2\text{ pts}}{=} 2N + \frac{3}{N}\left(\frac{N(N+1)}{2}\right)$

$= 2N + \frac{3}{2}(N+1) \overset{2\text{ pts}}{=} 3/2 + 7/2\, N.$ $\boxed{S_N = \frac{3}{2} + \frac{7}{2} N}$

(b) $\lim_{N\to\infty} \frac{1}{N} S_N = \lim_{N\to\infty}\left(\frac{3}{2N} + \frac{7}{2}\right) = 7/2.$

4 pts $\boxed{\lim_{N\to\infty} \frac{1}{N} S_N = \frac{7}{2}}$