Calculus I
Final

Common exa
11 December 20

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symboli manipulation capabilities. Please: 1) check answers when possible, 2) clearly indicate your answ and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*, 3 give exact answers, rather than decimal approximations to the answer.
Each question is followed by space to write your answer. Please write your solutions neatly in t space below the question. You are not expected to write each solution next to the statement of the question. If you use the back of a sheet, please indicate this by the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name KEY
Section
Last four digits of student identification number

1-10	11-20, 401
Grogan	Miker
Keech	Merrick
Mattingly	Taylor
Simon	Zeckner
Weaver	Boucher
Hislop	Freyer
Brown	Martin
Harris	Little

Question	Score	Total
p. 1/Q1–2		14
p. 2/Q3–4		14
p. 3/Q5–6		14
p. 4/Q7–8		14
p. 5/Q9–10		14
p. 6/Q11		14
p. 7/Q12		14
p. 8/Q13		14
Free	2	2
		100

(7)

1. Find the equation of the tangent line to the graph of $f(x) = x\cos(2x)$ at the point $x_0 = \pi/4$. Put your answer in the form $y = mx + b$ using exact values.

(+2) $f'(x) = \cos(2x) - 2x\sin(2x)$

(+1) $f(\frac{\pi}{4}) = \frac{\pi}{4}\cos(\frac{\pi}{2}) = 0$

(+1) $f'(\frac{\pi}{4}) = 0 - 2\cdot\frac{\pi}{4}\sin(\frac{\pi}{2}) = -\frac{\pi}{2}$

(+2) Tangent thru $(\pi/4, 0)$ with slope $-\frac{\pi}{2}$:

$y - 0 = (-\frac{\pi}{2})(x - \frac{\pi}{4})$

(+1) $\therefore y = -\frac{\pi}{2}x + \frac{\pi^2}{8}$

$y = -\frac{\pi}{2}x + \frac{\pi^2}{8}$

(7)

2. Evaluate the following limits and explain your computations.

(3) (a)

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2}$$

(4) (b)

$$\lim_{x\to -\infty} \frac{x+7}{\sqrt{4x^2+1}}$$

(a) $\lim_{x\to 2}\left(\frac{x^2-x-2}{x-2}\right) = \lim_{x\to 2}\frac{(x-2)(x+1)}{(x-2)} = \lim_{x\to 2}(x+1) = \boxed{3}$ by substitution rule

(+1) (+1) (+1)

(b) $\lim_{x\to -\infty}\frac{x+7}{\sqrt{4x^2+7}} = \lim_{x\to -\infty}\frac{x+7}{2|x|\sqrt{1+\frac{7}{4x^2}}} = \lim_{x\to +\infty}\frac{x(1+\frac{7}{x})}{2|x|\sqrt{1+\frac{1}{4x^2}}}$

Simplify +2
Answer +1
Reason +1

$= \lim_{x\to -\infty} -\frac{1}{2}\left(\frac{1+7/x}{\sqrt{1+1/4x^2}}\right) = -\frac{1}{2}$ by quotient law

(a) 3, (b) $-\frac{1}{2}$

⑦ 3. Suppose that a function $p(x)$ is defined as

$$p(x) = \begin{cases} x^2 + c & x < -1 \\ -1 + c^2 & x = -1 \\ c - x & x > -1 \end{cases}$$

④ (a) Find all values of c for which $\lim_{x \to -1} p(x)$ exists.

③ (b) Find all values of c for which $p(x)$ is continuous at $x = -1$.

(a) $\lim_{x \to -1^-} p(x) = (-1)^2 + c = 1 + c$ (+1) $\lim_{x \to -1^+} p(x) = \lim_{x \to -1^+} (c - x) = c + 1$ (+1)

(+2) $\lim_{x \to -1} p(x)$ exists if and only if $\lim_{x \to -1^-} p(x) = \lim_{x \to -1^+} p(x)$ $\therefore$ The limit exists for $c \in \mathbb{R}$

(b) For continuity, $\lim_{x \to -1} p(x) = p(-1)$ so we require $-1 + c^2 = c + 1$ (

$\therefore$ $c^2 - c - 2 = 0$ or $(c-2)(c+1) = 0$ $\therefore$ $c = -1$ or $c = 2$ (

(a) $c \in \mathbb{R}$, (b) $c = -1$, $c = +2$

⑦ 4. Let $f(x) = x^2 - 8$.

④ (a) Let $x_1, x_2, \ldots$ be the numbers which approximate a root of f in Newton's method. Find the formula for x_{n+1} in terms of x_n.

③ (b) If $x_1 = 3$, find x_2.

(a) (+1) $f'(x) = 2x$ Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f(x_n)}$ (+1)

(+2) $\therefore$ $x_{n+1} = x_n - \frac{x_n^2 - 8}{2x_n} = \frac{1}{2} x_n + \frac{4}{x_n}$ (not nece to simplify

(b) $x_2 = \frac{1}{2} \cdot 3 + \frac{4}{3} = \frac{3}{2} + \frac{4}{3} = \frac{17}{6}$ (+3)

(a) $x_{n+1} = x_n - \frac{x_n^2 - 8}{2x_n}$ (b) $x_2 = \frac{17}{6}$

⑦ 5. Let

$$F(x) = \int_x^{10} \frac{t}{t^2+4}\,dt.$$

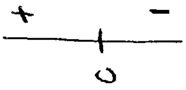
Find the intervals on which F is decreasing and increasing.

$$F'(x) = -\frac{x}{x^2+4}$$ (+3) (-1 if sign is wrong)

$F'(x) = 0$ if $x = 0$

$F'(x) > 0$ if $x < 0$

$F'(x) < 0$ if $x > 0$



(+4)

Interval(s) of increase $(-\infty, 0)$

Interval(s) of decrease $(0, \infty)$

⑦ 6. Find the absolute maximum value and absolute minimum value of

$$f(x) = \frac{x}{x^2+2}$$

on the interval $[0, 2]$. Justify your answer.

$$f'(x) = \frac{(x^2+2) - x(2x)}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2}$$ (+1)

Critical pts: $x = \pm\sqrt{2}$; only $x = +\sqrt{2}$ in $[0, 2]$ (+1)

$f(0) = 0$ $f(\sqrt{2}) = \frac{\sqrt{2}}{4}$ $f(2) = \frac{1}{3}$ (+1) Re

By closed interval method, extrema occur at endpts or critical points. Since $f(0) = 0$ is the min value, $f(0) = 0$ is the absolute minimum value. Since $f(\sqrt{2}) > f(2)$, $f(\sqrt{2}) = \frac{\sqrt{2}}{4}$ is the maximum value

Absolute maximum value $\sqrt{2}/4$ (+1),

Absolute minimum value 0 (+1)

⑦

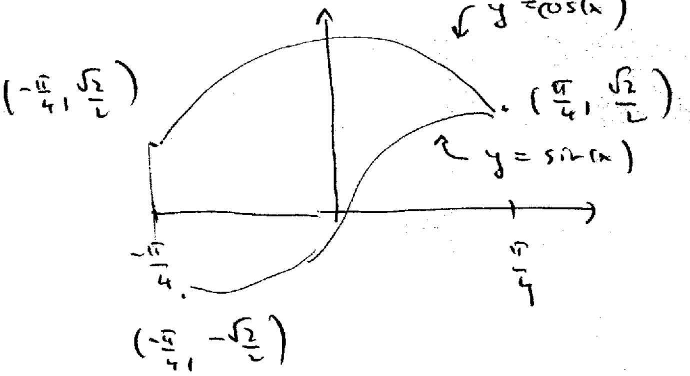
7. Let R be the region between the graphs of the functions $f(x) = \cos x$ and $g(x) = \sin x$ and with x in the interval $[-\pi/4, \pi/4]$. Sketch the region. Compute the area of R

Sketch

$$A = \int_{-\pi/4}^{\pi/4} [\cos(x) - \sin(x)]\, dx$$ (+3)

$$= \sin(x) + \cos(x) \Big|_{-\pi/4}^{\pi/4}$$ (+2)

$$= \sqrt{2}$$



$\sqrt{2}$

⑦

8. A rock is thrown upward from the top of a 25 meter tall building and falls to the ground. The initial speed of the rock is 20 meters/second. Assume that the acceleration of gravity is 10 meters/second^2 in the downward direction.

④ (a) Let $h(t)$ be the height above the ground in meters of the rock t seconds after the thrown. Find $h(t)$.

② (b) When does the rock hit the ground?

① (c) What is the velocity of the rock when it hits the ground?

(a) $h'(0) = 20\ \frac{m}{sec}$ $h''(0) = -10\ \frac{m}{sec}$ $h(0) = 25\ m$ (+1)

$h'(t) = -10t + C_1$ ∴ $C_1 = 20$ (+1)

$h(t) = -5t^2 + 20t + C_2$ ∴ $C_2 = 25$ (+1)

$h(t) = -5t^2 + 20t + 25$ (+1)

(b) $0 = -5t^2 + 20t + 25 = -5(t^2 - 4t - 5) = -5(t+1)(t-5)$ (+2)

∴ $t = 5$

(c) $h'(5) = -10 \cdot 5 + 20 = -30\ \frac{m}{sec}$ (+1)

(a) $h(t) = -5t^2 + 20t + 25$; (b) $t = 5$ sec,

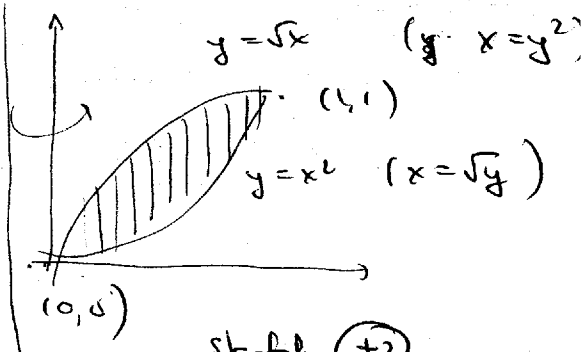
(c) -30 m/sec

⑦ 9. Let R be the region between the curves $y = \sqrt{x}$ and $y = x^2$. Sketch the region. Find the volume of the solid of revolution obtained by rotating R about the y-axis.

Intersection: $\sqrt{x} = x^2 \therefore x = x^4$

$\therefore x(x^3-1) = 0$

$\therefore x = 0$ or $x = 1$ (+1)



Sketch (+2)

Area (Washer Method):

(+4) $$V = \int_0^1 \pi\left((\sqrt{y})^2 - (y^2)^2\right) dy = \pi\int_0^1 (y - y^4)\,dy$$

$$= \pi(\tfrac{1}{2} - \tfrac{1}{5})$$

$$= \frac{3\pi}{10}$$

OR Shell method:

$$V = \int_0^1 2\pi x(\sqrt{x} - x^2)\,dx = 2\pi\int_0^1 (x^{3/2} - x^3)\,dx = 2\pi\left(\frac{2}{5} - \frac{1}{4}\right)$$

$$= 2\pi\,\frac{8-5}{20} = \frac{3\pi}{10}$$

$3\pi/10$

⑦ 10. Evaluate the following definite integrals. Be sure to show all your work.

③ (a) $\displaystyle\int_0^1 t\sqrt{1+4t^2}\,dt$ ④ (b) $\displaystyle\int_0^{\pi/4} \sec^2(\theta)\sin(\theta)\,d\theta$

(+3) (a) $u = 1 + 4t^2$ $du = 8t\,dt$

$$\therefore \int_0^1 t\sqrt{1+4t^2}\,dt = \frac{1}{8}\int_1^5 \sqrt{u}\,du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2}\Big|_1^5 = \frac{1}{12}(5^{3/2} - 1)$$

(+4) (b) $$\int_0^{\pi/4} \sec^2\theta \sin\theta\,d\theta = \int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta$$ $u = \cos\theta$, $du = -\sin\theta\,d\theta$

$$= -\int_1^{\sqrt{2}/2} \frac{du}{u^2} = \int_{\sqrt{2}/2}^1 \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_{\sqrt{2}/2}^1 = \sqrt{2} - 1$$

(a) $\frac{1}{12}(5^{3/2} - 1)$, (b) $\sqrt{2} - 1$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(14)

11. Let R be a rectangle with base on the x-axis and two corners on the graph of $y = 2 - x^4$ with $y \geq 0$.

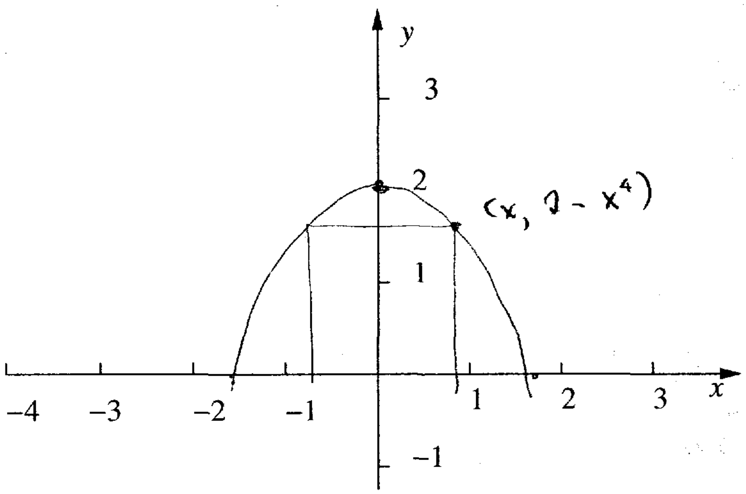
(4) (a) Graph the curve $y = 2 - x^4$ for $-2^{-1/4} \leq x \leq 2^{1/4}$. Sketch a typical rectangle R.

(3) (b) Find a function $A(x)$ that gives the area of R in terms of the x-coordinate of the upper right corner of R.

(4) (c) Use calculus to find the lengths of both of the sides of such a rectangle R with largest possible area.

(3) (d) Explain why you have found the largest possible area.

(+4) (a)



width: $2x$
Height: $2-x^4$

(+3) (b) $A(x) = (2x)(2-x^4) = 4x - 2x^5$ for $0 < x < 2^{1/4}$

(+4) (c) $A'(x) = 4 - 10x^4$ $A'(x) = 0$ if $10x^4 = 4$ or $x = \pm \left(\frac{2}{5}\right)^{1/4}$

$$\boxed{\begin{aligned} &\text{width} = 2x = 2\left(\tfrac{2}{5}\right)^{1/4} \\ &\text{Height} = 2 - x^4 = 2 - \tfrac{2}{5} = \tfrac{8}{5} \end{aligned}}$$

(+3) (d) Reason 1: $A''(x) = -10x^3 < 0$ on $[0, 2^{1/4})$;
∴ graph of $A(x)$ is concave down, so can have only 1 max

Reason 2: $A'(x)$ has sign graph

$$\begin{array}{ccccc} & + & & - & \\ 0 & \text{———} & \left(\frac{2}{5}\right)^{1/4} & \text{———} & \left(\frac{1}{2}\right)^{1/4} \end{array}$$

so $A\left(\left(\frac{2}{5}\right)^{1/4}\right)$ is global max

Reason 3: $A(x) = 0$ at each end pt, so by closed interval
∴ $A\left(\left(\frac{2}{5}\right)^{1/4}\right)$ is the absolute max.

⑭ (+4)

12. (a) Clearly state part 1 of the fundamental theorem of calculus.

(+4) (b) Let $f(x) = \begin{cases} 1, & x > 2 \\ 2, & x \leq 2 \end{cases}$ and define $F(x) = \int_2^x f(t)\,dt$.

Find the limits

$$\lim_{h\to 0+} \frac{F(2+h) - F(2)}{h} \quad \text{and} \quad \lim_{h\to 0^-} \frac{F(2+h) - F(2)}{h}.$$

③ (c) Explain why F is not differentiable at $x = 2$.

③ (d) Why does part 1 of the Fundamental Theorem of Calculus not imply that F is differentiable at $x = 2$?

(a) If f is continuous on $[a,b]$, (+1) and $g(x) = \int_a^x f(t)\,dt$ for $a \leq x \leq b$, (+1) then g is continuous on $[a,b]$, differentiable on (a,b), and $g'(x) = f(x)$ for $a < x < b$. (+1)

(b)

$$F(x) = \int_2^x 1\,dx = x-2 \qquad x>2$$

$$F(x) = \int_2^x 2\,dx = 2(x-2) \qquad x<2$$

$$\lim_{h\to 0^+} \frac{F(2+h)-F(2)}{h} = \lim_{h\to 0^+} \frac{(2+h)-2-(2-2)}{h} = \lim_{h\to 0^+} 1 = \boxed{1}$$

$$\lim_{h\to 0^-} \frac{F(2+h)-F(2)}{h} = \lim_{h\to 0^-} \frac{2(2+h-2)-2\cdot 0}{h} = \lim_{h\to 0^-} 2 = \boxed{2}$$

(c) $\lim_{h\to 0} \frac{F(2+h)-F(2)}{h}$ does not exist because left and right-hand limits don't agree. Hence, F is not differentiable at $x=2$.

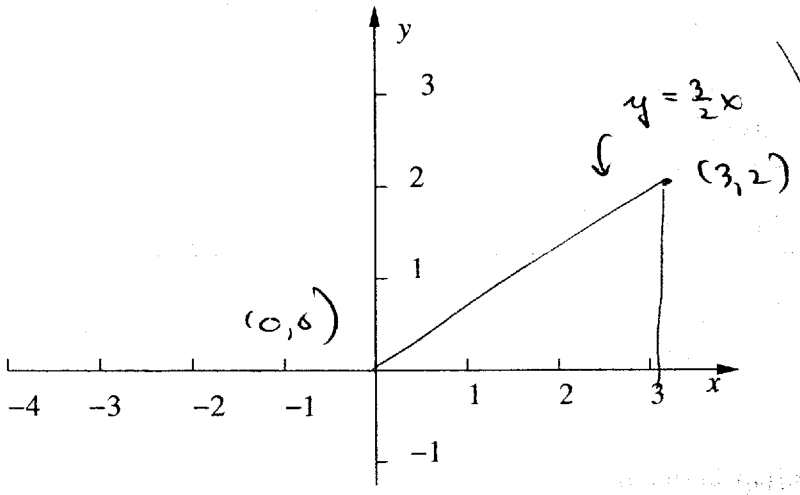
(d) The integrand $f(x)$ is not continuous at $x=2$.

13. 4 (a) Find a triangle T which is located in the first quadrant and so that when we revolve T about the x-axis, we obtain a right-circular cone with base a circle of radius 2 and with height 3. Sketch T on the axes below and give the coordinates of the three vertices of T.

6 (b) Write an integral for the volume of the right-circular cone obtained by rotating T about the x-axis.

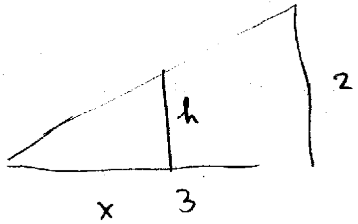
4 (c) Evaluate the integral in part (b) and find the volume of the cone.

(a)



+4

(b)



$\frac{x}{h} = \frac{3}{2} \Rightarrow h = \frac{2}{3}x$ (+2)

Cross-sectional area: $A(x) = \pi h^2 = \pi\left(\frac{2}{3}x\right)^2$ (+2)

$V = \int_0^3 A(x)\,dx = \int_0^3 \frac{4\pi x^2}{9}\,dx$ (+2)

(c) $V = \int_0^3 \frac{4\pi x^2}{9}\,dx = \frac{4\pi}{9}\,\frac{x^3}{3}\Big|_0^3 = \frac{4\pi}{9}\cdot\frac{27}{3} = \boxed{4\pi}$ (+4)