MA 113 — Calculus I | Fall 2008
Final Exam | December 18, 2008

Answer all of questions 1–7 and choose two of questions 8–10 to answer. Please indicate which of problems 8–10 is not to be graded by crossing through its number on the table below. At the end of the exam you find an Extra Credit Problem. This problem is optional and might earn you up to 10 bonus points.

Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. All other electronic devices including pagers and cell phones should be in the off position for the duration of the exam. Please:

1. **Clearly indicate your answer and the reasoning used to arrive at that answer. Unsupported answers may not receive credit.**
   You need not justify your answers to the Extra Credit Problem.
2. **Give exact answers, rather than decimal approximations to the answer.**

Each question is followed by space to write your answer. Please write your solutions neatly in the space provided.

Name: Answer Key

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		7
2		9
3		10
4		12
5		10
6		12
7		10
8		15
9		15
10		15
Total		**100**
Extra Credit		10

(1) Use calculus to find the absolute maximum and minimum values of $f(x) = xe^{-3x}$ on [0,1]. Also list the points where these extreme values occur.

(2) $f'(x) = e^{-3x} - 3xe^{-3x} = (1-3x)e^{-3x}$

(1) $f'(x) = 0$ for $x = \frac{1}{3}$

(2)

x	0	$\frac{1}{3}$	1
$f(x)$	0	$\frac{1}{3e}$	$\frac{1}{e^3}$

(2) abs. min abs. max

Absolute maximum value = $1/3e$ at x = $1/3$

Absolute minimum value = 0 at x = 0

(2) Given the function f defined for all x by

$$f(x) = \begin{cases} 2x+1 & \text{for } x < 3 \\ x^2 + Bx + 10 & \text{for } x \geq 3 \end{cases}$$

(a) Find B so that f is continuous for all x. **Show all limits that are needed to support your answer.**

(b) Use the value for B you found in (a) and determine all points x at which f is differentiable. **Indicate your reasoning.**

1 [(a) f is continuous on $(-\infty, 3)$ and $(3, \infty)$.

For $a = 3$

1 [$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x+1) = 7$ DSR

1 [$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 + Bx + 10) = 19 + 3B = f(3)$

1 [Wanted $19 + 3B = 7$, thus $\boxed{B = -4}$

1 [(b) f is diff$\underline{\text{able}}$ on $(-\infty, 3)$ and $(3, \infty)$

At $a = 3$:

3 [$\lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} = \frac{d}{dx}(2x+1)\Big|_{x=3} = 2$

$\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} = \frac{d}{dx}(x^2 - 4x + 10)\Big|_{x=3}$

$= (2x-4)\Big|_{x=3} = 2$

(a) $B =$ -4

Hence f is diff$\underline{\text{able}}$ at 3.

1 [(b) f is differentiable on the interval(s) $(-\infty, \infty)$

(3) Find the following limits. **Justify your steps in finding each limit.**

2

(a) $\lim_{t \to 0} \frac{2 + 6\sin(7t)}{2t+3} = $ $\frac{2}{3}$.

by direct substitution.

(b) $\lim_{x \to -\infty} x\, e^x = $ 0.

$= \lim_{x \to -\infty} \frac{x}{e^{-x}} \overset{\boxed{\frac{-\infty}{\infty}}}{=} \lim_{x \to -\infty} \frac{1}{-e^{-x}}$

l'Hopital

$= \lim_{x \to -\infty} -e^{x} = \underline{\underline{0}}$

(c) $\lim_{t \to 0} \frac{1 - \cos(t)}{t^2} = $ $\frac{1}{2}$.

$\overset{\boxed{\frac{0}{0}}}{=} \lim_{t \to 0} \frac{\sin(t)}{2t} \overset{\boxed{\frac{0}{0}}}{=} \lim_{t \to 0} \frac{\cos(t)}{2} = \frac{1}{2}$

l'Hospital

4

(4) Find the following derivatives. **Show your work!**

(a) $g'(x)$ when $g(x) = x^2 \tan(2x)$.

3

$$g'(x) = 2x\tan(2x) + x^2\sec^2(2x)\cdot 2$$
$$= 2x\left(\tan(2x) + x\sec^2(2x)\right)$$

(b) $\dfrac{dy}{dx}$, where $1 + xy^2 + x^3 = \sin(y)$. Your answer should be an expression involving x and y.

implicit differentiation:

3

$$y^2 + x\cdot 2y\,y' + 3x^2 = \cos(y)\,y'$$

2

$$y^2 + 3x^2 = (\cos(y) - 2xy)\,y'$$
$$y' = \frac{y^2+3x^2}{\cos(y) - 2xy}$$

(c) $f''(x)$ when $f(x) = \dfrac{4}{\sqrt{3+5x}}$. $= 4(3+5x)^{-\frac{1}{2}}$

2

$$f'(x) = -2(3+5x)^{-\frac{3}{2}}\cdot 5 = -10(3+5x)^{-\frac{3}{2}}$$

2

$$f''(x) = (-10)\left(-\tfrac{3}{2}\right)\cdot 5\,(3+5x)^{-\frac{5}{2}}$$
$$= 75(3+5x)^{-\frac{5}{2}}$$

(a) $g'(x) =$ $2x\left(\tan(2x) + x\sec^2(2x)\right)$

(b) $\dfrac{dy}{dx} =$ $\dfrac{y^2+3x^2}{\cos(y)-2xy}$

(c) $f''(x) =$ $75(3+5x)^{-\frac{5}{2}}$

(5) Consider the function $f(x) = \dfrac{x+2}{x-1}$.

(a) Compute the Riemann sum for f on the interval $[2,6]$ with $n = 4$ subintervals and the right endpoints as sample points.

2 3 4 5 6

width = 1

(1) $S_4 = 1 \cdot (f(3) + f(4) + f(5) + f(6))$

(1) $= \frac{5}{2} + \frac{6}{3} + \frac{7}{4} + \frac{8}{5} = \frac{157}{20} = 7.85$

(b) Show that f is decreasing on the interval $[2,6]$.

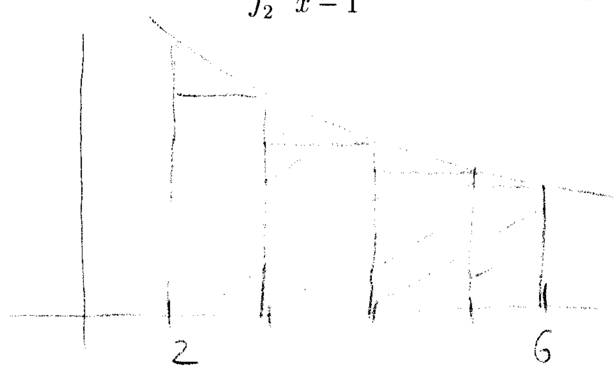
(2) $f'(x) = \dfrac{(x-1)-(x+2)}{(x-1)^2} = \dfrac{-3}{(x-1)^2}$

(2) $f'(x) < 0$ for all x in $[2,6]$.

Hence f is decreasing on $[2,6]$.

(c) Use (b) to determine whether the Riemann sum in (a) is greater or less than the actual value of the integral $\displaystyle\int_2^6 \frac{x+2}{x-1}\,dx$. **Do not compute this integral!**

(4)



f decreasing and positive on $[2,6]$

(a) Riemann sum = $\frac{157}{20} = 7.85$

(c) Riemann sum is less than the integral.

for correct and complete reasoning.

(6) Find the following integrals. **Show your work.**

2 (a) $\int \left(10x^4 - 3x^2 + 4e^x - 7\sin(x)\right) dx =$ $2x^5 - x^3 + 4e^x + 7\cos x + C$

(b) $\int_1^3 x^2 e^{x^3}\, dx =$ $\frac{1}{3}(e^{27} - e)$

2 $u = x^3, \quad du = 3x^2\, dx$

2 $\int_1^3 x^2 e^{x^3}\, dx = \frac{1}{3}\int_1^{27} e^u\, du = \frac{1}{3} e^u \Big|_1^{27}$

1 $= \frac{1}{3}(e^{27} - e)$

(c) $\int_1^x \frac{t^2+3}{t}\, dt =$ $\frac{1}{2}x^2 + 3\ln|x| - \frac{1}{2}$

1

$= \int_1^x t + \frac{3}{t}\, dt = \left(\frac{1}{2}t^2 + 3\ln|t|\right)\Big|_1^x$ 2

$= \frac{1}{2}x^2 + 3\ln|x| - \frac{1}{2}$

2

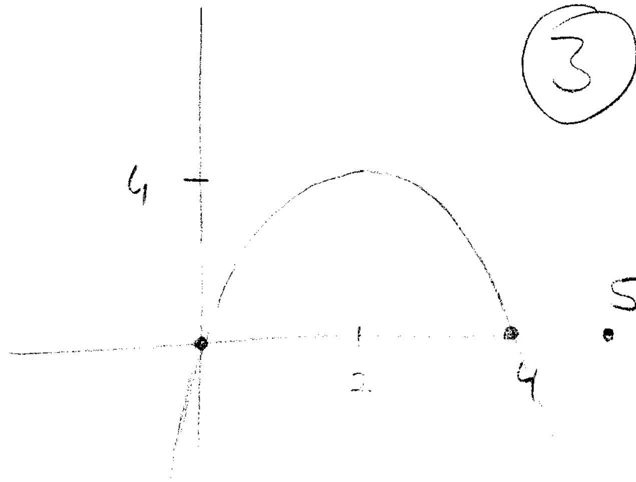
(7) A particle is traveling along a straight line so that its velocity at time t is given by $v(t) = 4t - t^2$ (measured in meters per second).

(a) Graph the function $v(t)$.

(b) Find the total distance traveled by the particle during the time period $0 \le t \le 5$.

(a) $v(t) = t(4-t)$

3



2 (b) $\displaystyle\int_0^5 |v(t)|\,dt$

2 $\displaystyle = \int_0^4 v(t)\,dt + \int_4^5 -v(t)\,dt$

2 $\displaystyle = \int_0^4 4t - t^2\,dt + \int_4^5 t^2 - 4t\,dt$

$\displaystyle = \left(2t^2 - \frac{1}{3}t^3\right)\Big|_0^4 + \left(\frac{1}{3}t^3 - 2t^2\right)\Big|_4^5$

1 $\displaystyle = \left(32 - \frac{64}{3}\right) - 0 + \left(\frac{125}{3} - 50\right) - \left(\frac{64}{3} - 32\right)$

$= 13$ meters

Total distance traveled = 13 meters

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) Let f be a function on the interval $[a, b]$. State the first part of the fundamental theorem of calculus. Do not forget to include the assumptions needed for f.

(4) let f be continuous on $[a,b]$. Put $g(x) = \int_a^x f(t)\,dt$ for x in $[a,b]$. Then g is differentiable on (a,b) and $g'(x) = f(x)$.

(b) Illustrate your answer in (a) by finding the derivative of $F(x) = \int_1^x (\ln t)^2\, dt$ at $x = e$.

(2) $F'(x) = (\ln(x))^2$

(1) $F'(e) = (\ln(e))^2 = 1$

(c) Find the derivative of the function $g(x) = \int_{x^2}^4 \sqrt{3 + 5t^4}\, dt$.

(2) $h(x) = \int_4^x \sqrt{3+5t^4}\,dt$, $h'(x) = \sqrt{3+5x^4}$

by FTC, Part 1. Then

(3) $g(x) = -\int_4^{x^2} \sqrt{3+5t^4}\,dt = -h(x^2)$ and

(3) $g'(x) = -h'(x^2)\cdot 2x = -2x\sqrt{3+5x^8}$.

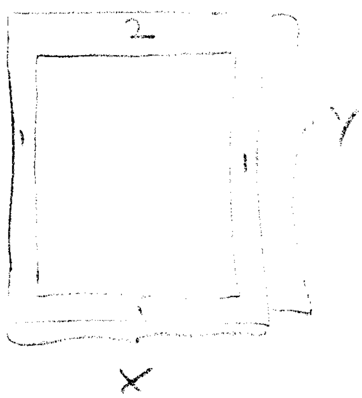
(b) $F'(e) =$ 1

(c) $g'(x) =$ $-2x\sqrt{3+5x^8}$

(9) A poster is to be made out of 150 in^2 cardboard and has to have 1-inch margins at the bottom and the two sides and a 2-inch margin at the top.

(a) Among all such posters what dimensions (width and height of the cardboard) will give the largest printed area?

(b) What is the largest printed area?



$xy = 150$

$y = \frac{150}{x}$ 2

2 (a) Printed area $= (x-2)(y-3)$

$A(x) = (x-2)\left(\frac{150}{x} - 3\right) = 150 - \frac{300}{x} - 3x + 6$

2 to be considered on $(0, \infty)$.

2 $A'(x) = \frac{300}{x^2} - 3 = \frac{300 - 3x^2}{x^2}$

2 $A'(x) = 0 \iff x^2 = 100 \iff x = 10$

3

	0		10		
$A'(x)$		+		−	
$A(x)$		↗		↘	

Hence $A(x)$ has an abs. max at $\boxed{x = 10}$.

Then $\boxed{y = 15}$

2 (b) $A(10) = 8 \cdot 12 = 96$

(a) Dimensions of the poster = 10 × 15 in.

(b) Largest printed area = 96 in^2.

(10) For this problem use the information that for a sphere with radius r

$$\text{the surface area is } 4\pi r^2 \text{ and the volume is } \frac{4}{3}\pi r^3.$$

A spherical snowball melts so that its surface area decreases at a rate of 1 cm^2/min.

(a) Find the rate at which the radius decreases at the moment when the radius is 3 cm.

(b) Find the rate at which the volume decreases at the moment when the radius is 3 cm.

2 (a) $S(t) = 4\pi\, r(t)^2$

1 $S'(t) = -1$

3 $-1 = S'(t) = 4\pi\, 2\, r(t)\, r'(t) = 8\pi\, r(t)\, r'(t).$

At time t_0 when $r(t_0) = 3$ we get

3 $-1 = 8 \cdot \pi \cdot 3 \cdot r'(t)$,

$r'(t) = \dfrac{-1}{24\pi}$ cm/min

2 (b) $V(t) = \dfrac{4}{3}\pi\, r(t)^3$

2 $V'(t) = 4\pi\, r(t)^2\, r'(t)$

At time t_0 as in (a) we get

2 $V'(t_0) = 4\pi \cdot 3^2 \cdot \dfrac{-1}{24\pi} = -\dfrac{3}{2}$ cm³/min

(a) Radius decreases at a rate of $\dfrac{-1}{24\pi}$ cm/min.

(b) Volume decreases at a rate of $-\dfrac{3}{2}$ cm^3/min.

Extra Credit Problem.
Circle the correct answer. For each correct answer you earn 2 points, and for each incorrect answer 1 point will be subtracted. Therefore, **it might be wise to skip a question rather than risking to lose a point.** However, your final score on this problem will not be negative! You need not justify your answer.

True	**False**	**Statement**
☐	☒	$\frac{d}{dx}(5^x) = x5^{x-1}$.
☒	☐	The function $f(x) = \int_1^x \lvert 2t-1 \rvert \, dt$ is differentiable on $(-\infty, \infty)$.
☐	☒	If the function f is continuous on the interval $[a, b]$, then f is differentiable on the interval (a, b).
☒	☐	If the function f is differentiable and $f(-1) = f(1)$, then there exists a number c such that $\lvert c \rvert < 1$ and $f'(c) = 0$.
☒	☐	$\int_{-\pi}^{\pi} \sin(x^3)\, dx = 0$.

odd function

Use MVT (or Rolle's Thm.) for the interval $[-1, 1]$.