MA 113 — Calculus I **Fall 2009**
Exam 4 **December 15, 2009**

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

(a) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,

(b) give exact answers, rather than decimal approximations to the answer(unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Key

Section: ____________________

Last four digits of student identification number: floor[frac[π^e]x10^4]

Question	Score	Total
1		9
2		13
3		8
4		6
5		12
6		9
7		10
8		15
9		15
10		15
Free	3	3
		100

1. Find the following limits:

(a) $\lim_{x\to 0}\dfrac{2\sin x-\sin(2x)}{5x}$

(b) $\lim_{x\to\infty}\dfrac{\ln\left(2+e^{2x}\right)}{5x}$

(c) $\lim_{x\to 0}\dfrac{e^{2x}-1}{\tan x}$

(a)

$$\lim_{x\to 0}\frac{2\sin x-\sin(2x)}{5x}\underset{=}{H}\lim_{x\to 0}\frac{2\cos x-2\cos(2x)}{5}$$
$$=0$$

$2:\begin{cases}1: \text{both derivatives correct}\\ 1: \text{answer}\end{cases}$

(b)

$$\lim_{x\to\infty}\frac{\ln\left(2+e^{2x}\right)}{5x}\underset{=}{H}\lim_{x\to\infty}\frac{\frac{2e^{2x}}{2+e^{2x}}}{5}=\lim_{x\to\infty}\frac{2e^{2x}}{10+5e^{2x}}$$
$$\underset{=}{H}\lim_{x\to\infty}\frac{4e^{2x}}{10e^{2x}}=\frac{2}{5}$$

$4:\begin{cases}1: \text{both derivatives correct (1st application)}\\ 1: \text{algebra}\\ 1: \text{both derivatives correct (2nd application)}\\ 1: \text{answer}\end{cases}$

(c) $\lim_{x\to 0}\dfrac{e^{2x}-1}{\tan x}\underset{=}{H}\lim_{x\to 0}\dfrac{2e^{2x}}{\sec^2 x}=2$

$2:\begin{cases}1: \text{both derivatives correct}\\ 1: \text{answer}\end{cases}$

1: mention at some point the applicability of l'Hospital's Rule

(a) $\lim_{x\to 0}\dfrac{2\sin x-\sin(2x)}{5x}=$ ___0___

(b) $\lim_{x\to\infty}\dfrac{\ln\left(2+e^{2x}\right)}{5x}=$ ___2/5___

(c) $\lim_{x\to 0}\dfrac{e^{2x}-1}{\tan x}=$ ___2___

Total: 9

2. Consider the function $g(x) = x^3 - 6x^2 + 9x + 2$ on the closed interval $[-1,4]$.

(a) List all of the critical points of $g(x)$.

(b) On what intervals is $g(x)$ increasing?

(c) On what intervals is $g(x)$ concave down?

(d) List all x- and y-coordinates for the absolute maximum.

(e) List all x- and y-coordinates for the absolute minimum.

(a) $g'(x) = 3x^2 - 12x + 9$ $g'(x) = 0$ $3(x-1)(x-3) = 0$ $x = 1,3$	$2: \begin{cases} 1: g'(x) \text{ correct} \\ 1: \text{both critical points} \end{cases}$
(b) The function is increasing on the intervals $-1 < x < 1$ and $3 < x < 4$ because the $g' > 0$ there.	$3: \begin{cases} 2: (-1,1) \cup (3,4) \\ 1: \text{reason} \end{cases}$
(c) $g''(x) = 6x - 12$, so $g(x)$ is concave down on $(-1,2)$, since $g'' < 0$ there.	$2: \begin{cases} 1: \text{2nd derivative correct} \\ 1: \text{answer and reason} \end{cases}$
(d) Evaluate $g(-1) = -14$, $g(1) = 6$ and $g(4) = 6$. The absolute maximum occurs at $(1,6)$ and at $(4,6)$.	$4: \begin{cases} 2: (1,6) \\ 2: (4,6) \end{cases}$
(e) Evaluate $g(-1) = -14$, $g(3) = 2$ and $g(4) = 6$. The absolute minimum occurs at $(-1,-14)$.	2: $(-1,-14)$

Total: 13

(a) Critical points: x = 1,3

(b) $g(x)$ is increasing on $(-1,1) \cup (3,4)$

(c) $g(x)$ is concave down on $(-1,2)$

(d) The absolute maximum occurs at the point(s): (1,6) and (4,6)

(e) The absolute minimum occurs at the point(s): $(-1,-14)$

(c) Consider the curve $y^2 + xe^y = 1$.

(a) Find the derivative, $\frac{dy}{dx}$, of y.

(b) Find the slope of the tangent line to this curve at the point (1,0).

(c) Find the equation of the tangent line at the point (1,0). Express it in the form $y = mx + b$.

Solution	Scoring
(a) $y^2 + xe^y = 1$ $2y\frac{dy}{dx} + e^y + xe^y\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{e^y}{2y + xe^y}$	$4: \begin{cases} 1: \text{use implicit differentiation} \\ 1: d(y^2) = 2yy' \\ 1: \text{product rule and } d(xe^y) \\ 1: \text{answer} \end{cases}$
(b) $m = \left.\frac{dy}{dx}\right\vert_{x=1, y=0} = -\frac{e^0}{2(0) + 1\cdot e^0} = -1$	$2: \begin{cases} 1: \text{evaluate derivative at } (1,0) \\ 1: \text{answer} \end{cases}$
(c) $y - 0 = -1(x - 1)$ $y = -x + 1$	$2: \begin{cases} 1: \text{use point } (1,0) \text{ and slope } -1 \\ 1: \text{answer} \end{cases}$

(a) $\frac{dy}{dx} = -\frac{e^y}{2y + xe^y}$

(b) The slope of the tangent line: $m =$ -1

(c) The equation of the tangent line: y = -x + 1

Total: 8

4.

(a) If $f(x)=x^{10}g(x)$, $g(1)=2$, and $g'(1)=3$, find $f'(1)$.

(b) If $h(x)=\cos\left(x^2+e^{x^2}\right)$, find $h'(x)$.

(a) $f(x) = x^{10}g(x)$

$f'(x)=10x^9g(x)+x^{10}g'(x)$

$f'(1)=10\times(1)^9\times g(1)+1^{10}\times g'(1)$

$=20+3=23$

$3:\begin{cases}1: \text{correct derivative}\\ 1: \text{substitute } x=1\\ 1: \text{answer}\end{cases}$

(b) $h(x)=\cos\left(x^2+e^{x^2}\right)$

$h'(x)=-\sin(x^2+e^{x^2})(2x+2xe^{x^2})$

$3:\begin{cases}1: \text{derivative of outside function}\\ 1: \text{derivative of inside function}\\ 1: \text{answer}\end{cases}$

(a) $f'(1)=$ 23

(b) $h'(x)=$ $-\sin(x^2+e^{x^2})(2x+2xe^{x^2})$

Total: 6

5. Find the following integrals and/or antiderivatives. You must show all of your work to receive full credit. An answer without supporting work will receive no credit.

(a) $\int_1^2 \left(10x^4 - 4x^3 - 2x - \frac{2}{x^2} + 2 \right) dx$

(b) $\int_1^x \frac{t^2 \cos t + 4}{t^2} dt$

(c) $\int x \sin(x^2)\, dx$

(d) $\int_0^4 \frac{x}{x+9} dx$.

(a) $\int_1^2 \left(10x^4 - 4x^3 - 2x - \frac{2}{x^2} + 2\right) dx =$

$$\left(2x^5 - x^4 - x^2 + \frac{2}{x} + 2x\right|_1^2 = 45$$

$3:\begin{cases} 1: \text{2 of 5 antiderivatives correct} \\ 1: \text{3 of 5 antiderivatives correct} \\ 1: \text{answer} \end{cases}$

(b)

$$\int_1^x \frac{t^2 \cos t + 4}{t^2} dt = \int_1^x \left(\cos t + \frac{4}{t^2}\right) dt$$

$$= \left(\sin t - \frac{4}{t}\right|_1^x = \sin x - \frac{4}{x} - \sin(1) + 4$$

$3:\begin{cases} 1: \text{simplify and 1 antiderivative} \\ 1: \text{2nd antiderivative} \\ 1: \text{answer} \end{cases}$

(c) $\int x \sin(x^2)\, dx = \frac{1}{2}\int \sin u \; du \qquad u = x^2$

$$= -\frac{1}{2}\cos u + C$$

$$= -\frac{1}{2}\cos(x^2) + C$$

$3:\begin{cases} 1: \text{a reasonable substitution} \\ 1: \text{correct antiderivative} \\ 1: \text{answer} \end{cases}$

(d) $\int_0^4 \frac{x}{x+9} dx = \int_9^{13} \frac{u-9}{u} du \qquad u = x + 9$

$$= \int_9^{13} \left(1 - \frac{9}{u}\right) du = \left(u - 9\ln(u)\right|_9^{13}$$

$$= 4 - 9\ln(13) + 9\ln(9)$$

$$\approx 0.6905$$

Other acceptable answers: 4+9 ln(9/13); 4+ln(387420489/10604499373); 4+ln(0.036534);

$3:\begin{cases} 1: \text{reasonable substitution} \\ 1: \text{correct antiderivative} \\ 1: \text{answer} \end{cases}$

If a change of variables is done incorrectly, take the point from the answer point.

(a) $\int_1^2 \left(10x^4 - 4x^3 - 2x - \frac{2}{x^2} + 2\right) dx =$ 45

(b) $\int_1^x \frac{t^2 \cos t + 4}{t^2} dt =$ $\sin x - \frac{4}{x} - \sin(1) + 4$

(c) $\int x \sin(x^2)\, dx =$ $-\frac{1}{2}\cos(x^2)+C$

(d) $\int_0^4 \frac{x}{x+9} dx =$ 4 + 9ln(9) - 9ln(13)

Total: 12

6. Consider the function $F(x)=\int_0^x \frac{t}{1+t^4}dt$.

(a) Find all intervals on which $F(x)$ is increasing.

(b) Find all intervals on which $F(x)$ is concave up.

(a) $F'(x)=\frac{x}{1+x^4}$. $F'(x)>0$ for $x > 0$, so F is increasing for $x > 0$.	$4:\begin{cases}1: \text{derivative}\\1: \text{set derivative} > 0\\1: \text{answer}\\1: \text{reason}\end{cases}$
(b) $F''(x)=\frac{1+x^4-x(4x^3)}{(1+x^4)^2}=\frac{1-3x^4}{(1+x^4)^2}$ $F''(x)=0$ for $x=\pm 1/\sqrt[4]{3}$. F is concave up on $\left(-\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$ since $F''(x) > 0$ there.	$5:\begin{cases}1: F''(x)\\1: \text{deal with inflection points}\\1: \text{find interval}\\1: \text{answer}\\1: \text{reason}\end{cases}$

(a) $F(x)$ is increasing on $(0,\infty)$ or for $x > 0$

(b) $F(x)$ is concave up on $\left(-\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$

Total: 9

7. For this problem use the following information about a basketball.

$$\text{Surface Area: } A = 4\pi r^2 \qquad \text{Volume: } V = \frac{4}{3}\pi r^3.$$

A basketball is being inflated and its volume is increasing at the rate of 5 cm³/sec. Be sure to remember to include units in your response.

(a) Find the rate at which the radius, r, of the ball is changing when the radius is 5 cm.

(b) Is the radius increasing or decreasing when the radius is 5 cm? Justify your answer.

(c) Find the rate at which the surface area is changing when the radius is 5 cm.

(d) Is the surface area increasing or decreasing when the radius is 5 cm? Justify your answer.

(a) $\frac{dV}{dt} = 5 \quad cm^3 / sec$

$4\pi r^2 \frac{dr}{dt} = \frac{dV}{dt} = 5$

$\frac{dr}{dt} = \frac{5}{4\pi r^2}$

$\left.\frac{dr}{dt}\right|_{r=5} = \frac{5}{4\pi 5^2} = \frac{1}{20\pi} cm / sec$

$4 : \begin{cases} 1 : \frac{dV}{dt} = 5 \\ 1 : \text{find correct expression for dV/dt} \\ 1 : \text{find correct expression for dr/dt} \\ 1 : \text{answer} \end{cases}$

(b) $\left.\frac{dr}{dt}\right|_{r=5} > 0$ so the radius is increasing when the radius is 5 cm.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\left.\frac{dS}{dt}\right|_{r=5} = 8\pi \times 5 \times \frac{1}{20\pi}$

$\left.\frac{dS}{dt}\right|_{r=5} = 2 \ cm^2 / sec$

$2 : \begin{cases} 1 : dS/dt \\ 1 : \text{answer} \end{cases}$

(d) $\left.\frac{dS}{dt}\right|_{r=5} > 0$ so the surface area is increasing when the radius is 5 cm.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

-1 point if units are not correct in (a) and (c).

Total: 10

Work two of the following three problems. Indicate the problems that is not to be graded by crossing through its number on the front of the exam.

8. (a) State the Mean Value Theorem. Use complete sentences.
   (b) Determine the values for the constants a and b such that the function f defined by

$$f(x)=\begin{cases}1 & x=0\\ ax+b & 0<x\le 1\\ x^2+4x+2 & 1<x\le 3\end{cases}$$

   satisfies all of the hypotheses of the Mean Value Theorem on the interval [0,3]. **As usual, show your work to support your answer.**
   (c) Find at least one point in [0,3] where the conclusion of the theorem is satisfied.

(a) Let f be a function that satisfies the following hypotheses:
(1) f is continuous on [a,b];
(2) f is differentiable on (a,b)
Then there is a number $c\in(a,b)$ so that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$$5:\begin{cases}1:\text{continuity hypothesis}\\ 1:\text{differentiability hypothesis}\\ 1:\text{conclusion}\\ 2:\text{completeness of statement}\end{cases}$$

(b) The function must be continuous [0,3] and differentiable on (0,3). Thus: $\lim_{x\to0^+} f(x)=f(0)=1$ so b = 1. F must be differentiable (hence continuous) at x = 1, so we must have that the derivative from the left $f'_-(1)=a$ be the same as the derivative from the right: $f'_+(1)=2(1)+4=6$. Thus a = 6.

$$5:\begin{cases}1:\text{discuss continuous and differentiable}\\ 1:b=1\\ 1:\text{left derivative is a}\\ 1:\text{right derivative is 6}\\ 1:\text{a=right derivative}\end{cases}$$

(c) $\dfrac{f(3)-f(0)}{3-0}=\dfrac{22}{3}$. For f to satisfy the MVT it must do so in the interval from 1 to 3. f'(x)=2x + 4 = 22/3 gives x = 5/3 (which does lie in the necessary interval.).

$$5:\begin{cases}2:\text{find slope of the secant line}\\ 2:x = 5/3\\ 1:\text{check that answer in correct interval}\end{cases}$$

Total: 15

9. (a) State both parts of the Fundamental Theorem of Calculus. Use complete sentences.

(b) Consider the function f on $[1,\infty)$ defined by $f(x)=\int_1^x \sin^2(u^2)\,du$. Explain why the function $f(x)$ is increasing.

(c) Find the derivative of the function $g(x)=\int_{x^3}^1 \sin^2(u^2)\,du$.

Solution	Scoring
(a) Suppose f is continuous on [a,b]. (I) If $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$. (II) $\int_a^b f(x)\,dx = F(b)-F(a)$, where F is any antiderivative of f, that is, F′ = f. (page 387)	$6:\begin{cases}3: \text{FTC I}\\ 3: \text{FTC II}\end{cases}$
(b) Since $\sin^2(x^2)$ is continuous and differentiable by the FTC I, $f'(x)=\sin^2(x^2)=(\sin(x^2))^2 \geq 0$ for all $x \geq 0$. Thus, f is increasing there.	$4:\begin{cases}2: f'(x)\\ 1: \text{reason}\\ 1: \text{reason FTC applies}\end{cases}$
(c) $g(x) = -f(x^3)$ by the properties of the integral. Thus, $g'(x) = -f'(x^3)\times(3x^2) = -3x^2\sin^2(x^6)$	$5:\begin{cases}2: \text{recognizing } g(x) = -f(x^3)\text{, somehow}\\ 2: \text{correct derivative}\\ 1: \sin^2(x^6) \text{ and not } \sin^2(x^5)\end{cases}$

Total: 15

10. A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by

$$v(t) = 3t^2 - 2t - 1 \text{ miles/minute.}$$

The position $x(t)$ is 5 for $t = 2$.

(a) Find the acceleration of the particle at time $t = 3$.
(b) Is the speed of the particle increasing at time $t = 3$? Give a reason for your answer.
(c) Find all values of t at which the particle changes direction. Justify your answer.
(d) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

(a) $a(t) = v'(t)$

$a(t) = 6t - 2$

$a(3) = 16 \quad \text{miles / minute}^2$

$2: \begin{cases} 1: a(t) \\ 1: a(3) \end{cases}$

(b) At t = 3, v(3) = 20 > 0 and a(3) > 0, so the speed of the particle is increasing.

$2: \begin{cases} 1: v(3) > 0 \\ 1: \text{reason} \end{cases}$

(c) The particle changes direction when the velocity changes sign.
$v(t) = 0 = 3t^2 - 2t - 1 = (3t + 1)(t - 1)$, so v(t) changes sign at t = 1 and at t = -1/3. -1/3 is not in the domain, so must be omitted.

$4: \begin{cases} 1: v(t)=0 \\ 1: t = 1 \\ 1: \text{check that } v(t) \text{ changes sign} \\ 1: \text{must deal with ommiting } t=-1/3 \end{cases}$

(d) Between 0 and 3 the velocity changes sign at t = 1, so the total distance traveled is:

$$\left|\int_0^1 v(t)dt\right| + \left|\int_1^3 v(t)dt\right| = \left|\int_0^1 (3t^2 - 2t - 1)dt\right| + \left|\int_1^3 (3t^2 - 2t - 1)dt\right|$$

$$= \left|(t^3 - t^2 - t)_0^1\right| + \left|(t^3 - t^2 - t)_1^3\right|$$

$$= |-1| + |16| = 17 \text{ miles}$$

$6: \begin{cases} 1: \text{integral} \\ 2: \text{integrate right parts} \\ 2: \text{correct subintegrals} \\ 1: \text{answer} - \text{must be positive} \end{cases}$

1: units correct in (a) and (d)

Total: 15