(1) Determine the following limits. Justify your steps in finding each limit.

(a) $\displaystyle\lim_{x\to 2}\frac{\sin(3x-6)}{x-2}$, (b) $\displaystyle\lim_{x\to\infty}(\sqrt{x-4}-\sqrt{x+2})$, (c) $\displaystyle\lim_{x\to\infty} e^{-x}\ln x$.

(a) This is of type $\frac{0}{0}$. Thus, we can use de l'Hospital's rule and obtain

$$\lim_{x\to 2}\frac{\sin(3x-6)}{x-2} = \lim_{x\to 2}\frac{3\cos(3x-6)}{1} = 3\cos(0) \quad \text{by continuity}$$
$$= \underline{\underline{3}}.$$

(b)
$$\lim_{x\to\infty}\left(\sqrt{x-4}-\sqrt{x+2}\right) = \lim_{x\to\infty}\frac{x-4-(x+2)}{\sqrt{x-4}+\sqrt{x+2}} \quad \text{(expansion by the conjugate)}$$
$$= \lim_{x\to\infty}\frac{-6}{\sqrt{x-4}+\sqrt{x+2}} = \underline{\underline{0}} \quad \text{because}$$

$$\lim_{x\to\infty}\sqrt{x-4} = \lim_{x\to\infty}\sqrt{x+2} = \infty.$$

(c)
$$\lim_{x\to\infty} e^{-x}\cdot\ln x = \lim_{x\to\infty}\frac{\ln x}{e^x}, \quad \text{which is of type } \frac{\infty}{\infty}.$$

Hence, de l'Hospital's rule provides

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty}\frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty}\frac{1}{x\cdot e^x} = \underline{\underline{0}}.$$

(a) $\displaystyle\lim_{x\to 2}\frac{\sin(3x-6)}{x-2} = \underline{3}$, (b) $\displaystyle\lim_{x\to\infty}(\sqrt{x-4}-\sqrt{x+2}) = \underline{0}$, (c) $\displaystyle\lim_{x\to\infty} e^{-x}\ln x = \underline{0}$

(2) For a constant c, consider the function f defined by

$$f(x) = \begin{cases} \ln(x^2+1)+2 & \text{if } x \le 0 \\ c\sqrt{x+4} & \text{if } x > 0 \end{cases}.$$

(a) Find c such that f is continuous at $x=0$. Show all limits that are needed to support your answer.

(b) Use the value for c you found in (a) and determine all points x at which f is differentiable. Indicate your reasoning.

(a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (\ln(x^2+1)+2) = \ln(1)+2$ (by continuity)
$= 2.$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} c\sqrt{x+4} = 2c$ by continuity. Hence $\lim_{x\to 0} f(x)$ exists if and only if $2 = 2c$, i.e. $c=1$. Then $\lim_{x\to 0} f(x) = 2 = f(0)$, so f is indeed continuous if $c=1$.

(b) $\ln(x^2+1)+2$ and $\sqrt{x+4}$ are differentiable functions. Thus, f is differentiable at all $x \neq 0$. At $x=0$ we consider the one-sided limits.

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{\ln(x^2+1)+2-2}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0^-} \frac{\frac{2x}{x^2+1}}{1}$$

by de l'Hospital's rule or by def. of the derivative

$= 0$ by continuity.

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{\sqrt{x+4}-2}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0^-} \frac{\frac{1}{2}(x+4)^{-\frac{1}{2}}}{1}$$

continuity $\rightarrow = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$

The one-sided limits do not agree, thus $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x}$ does not exist.

(a) $c =$ <u>1</u>, (b) f is differentiable on $\{x \mid x \neq 0\} = (-\infty,0)\cup(0,\infty)$

(3) Suppose that f is a differentiable function, that $f'(x) \geq 2$ for all $x \geq 1$, and that $f(1) = 5$. Use the Mean Value Theorem to argue that $f(x) \geq 2x + 3$ for all $x \geq 1$.

1) This is true if $x = 1$ because $f(1) = 5 \geq 2 \cdot 1 + 3 = 5$.

If $x > 1$, then the MVT gives the existence of some c in the open interval $(1, x)$ such that

$$\frac{f(x) - f(1)}{x - 1} = f'(c).$$

Since $c > 1$, the assumption yields $f'(c) \geq 2$, thus $\frac{f(x) - f(1)}{x-1} \geq 2$.

Solving for $f(x)$ we obtain $f(x) - 5 \geq 2(x-1)$, thus

$$f(x) \geq 2x - 2 + 5 = 2x + 3,$$ as claimed.

2)

(4) Find the absolute minimum value of $f(x) = x^2 + \dfrac{16}{x^2}$ on the interval $(0, \infty)$. Justify your answer and list also the point(s) where the extremal value occurs.

$f'(x) = 2x - \dfrac{32}{x^3}$, which is zero if and only if $2x^4 = 32$, i.e. $x^4 = 16$.

Since $x > 0$, this implies $x = 2$. Using the table

x	$(0, 2)$	2	$(2, \infty)$
sign of $f'(x)$	$-$		$+$
I/D	$\searrow$		$\nearrow$

we conclude that f is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$. Hence f assumes its absolute minimum value at 2 and it is $f(2) = 2^2 + \dfrac{16}{2^2} = 4 + 4 = \underline{\underline{8}}$.

Absolute minimum value = 8 at $x =$ 2

(5) Consider the function $f(x) = \dfrac{x}{x-1}$.

(a) Compute the Riemann sum for f on the interval $[2, 6]$ with $n = 4$ subintervals and the left endpoints as sample points. Give the precise result as a rational number.

(b) Show that f is decreasing on the interval $[2, 6]$.

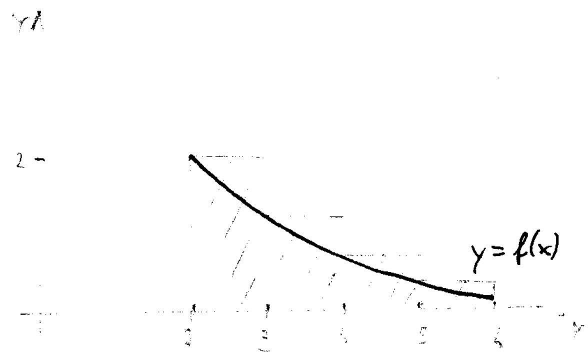
(c) Without computing the integral $\int_2^6 f(x)dx$, decide whether the Riemann sum in (a) is greater or less than this integral.

(a) The width of the intervals is $\frac{6-2}{4} = 1$. Thus the Riemann sum is

$$S_4 = 1 \cdot [f(2) + f(3) + f(4) + f(5)] = 2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} = \frac{24+18+16+15}{12} = \frac{73}{12}.$$

(b) $f'(x) = \dfrac{x-1-x}{(x-1)^2} = -\dfrac{1}{(x-1)^2} < 0$ if $x > 1$. Hence f is decreasing on $[2, 6]$.

(c)



f is positive and decreasing on $[2,6]$. Thus the Riemann sum S_4 overestimates the area under the curve $y = f(x)$.

(a) Riemann sum = $\frac{73}{12}$

(c) Riemann sum is larger than the integral.

(6) Determine the following integrals. Show your work.

(a) $\int(12x^5 - 4\cos x - 5e^x)dx$, (b) $\int_1^x \frac{t\sin t + 2}{t}dt$, (c) $\int_0^1 x^4 e^{-x^5}dx$.

(a) $\int(12x^5 - 4\cos x - 5e^x)dx = \underline{\underline{2x^6 - 4\sin x - 5e^x + C}}.$

(b) $\int_1^x \frac{t\cdot\sin t + 2}{t}dt = \int_1^x \left(\sin t + \frac{2}{t}\right)dt = \left(-\cos t + 2\ln t\right)\Big|_1^x$

$= \underline{\underline{-\cos x + 2\ln x + \cos(1)}}.$

(c) Substitute $u = -x^5$, so $du = -5x^4\cdot dx$. Then

$\int_0^1 x^4 e^{-x^5}dx = \int_0^{-1} -\frac{1}{5}e^u du = -\frac{1}{5}e^u\Big|_0^{-1} = \underline{\underline{-\frac{1}{5}\left(\frac{1}{e}-1\right)}}.$

(a) $\int(12x^5 - 4\cos x - 5e^x) = \underline{2x^6 - 4\sin x - 5e^x + C}$,

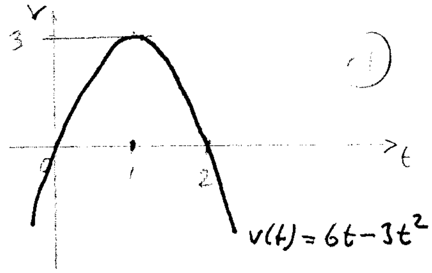
(b) $\int_1^x \frac{t\sin t + 2}{t}dt = \underline{-\cos x + 2\ln x + \cos(1)}$, (c) $\int_0^1 x^4 e^{-x^5}dx = \underline{-\frac{1}{5}\left(\frac{1}{e}-1\right)}$

(7) A particle is traveling along a straight line so that its velocity at time t is given by $v(t) = 6t - 3t^2$ measured in meters per second.

(a) Sketch the graph of v. Be sure to mark the t-intercepts.

(b) Find the total distance traveled by the particle during the time period $0 \leq t \leq 3$.

(a)



1

$v(t) = 3t(2-t) = 0$ if and only if $t=0$ or $t=2$.

2

(b) The total distance is

$$\int_0^3 |v(t)|\,dt = \int_0^2 v(t)\,dt + \int_2^3 -v(t)\,dt$$

2

$$= \int_0^2 (6t-3t^2)\,dt + \int_2^3 (3t^2-6t)\,dt$$

2

$$= (3t^2 - t^3)\Big|_0^2 + (t^3 - 3t^2)\Big|_2^3$$

$$= 12 - 8 + 27 - 27 - (8-12)$$

$$= \underline{\underline{8}}.$$

Total distance traveled = 8 meters.

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) Define what it means for a function f to be continuous at a. Use complete sentences.

(b) Let c be a number and consider the function

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x} - c & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ c\ln(e + x) - 4 & \text{if } x > 0 \end{cases}$$

Find all numbers c such that $\lim_{x\to 0} f(x)$ exists.

(c) Is there a number c such that the function f in part (b) is continuous at 0? As always, justify your answer.

4

(a) The function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(\frac{\cos x - 1}{x} - c\right) = -c + \lim_{x\to 0^-} \frac{\cos x - 1}{x}$

de l'Hospital $\longrightarrow$ $\overset{\frac{0}{0}}{=} -c + \lim_{x\to 0^-} \frac{-\sin x}{1} = -c + 0 = -c$ by continuity.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (c\ln(e+x) - 4) = c\cdot\ln(e) - 4 = c - 4$ by continuity.

2

Hence $\lim_{x\to 0} f(x)$ exists if and only if $-c = c - 4$, that is, $4 = 2c$, so $\underline{c = 2}$.

(c) If $c = 2$, then $\lim_{x\to 0} f(x) = -c = -2 \neq 2 = f(0)$.

2

Hence f is never continuous at 0.

(b) $c =$ 2 (c) yes / no (*circle the correct answer*)

(9) (a) State the Chain rule. Use complete sentences.

(b) Consider the curve described by the equation $\ln(y^2 - 3) = xy - 2$. Find the equation of the tangent line to this curve at the point $(1, 2)$. Write your answer in the form $y = mx + b$. As always, show your work.

(a) If g is differentiable at a and f is differentiable at $g(a)$, then the function $f \circ g$ is differentiable at a and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

(b) Let $y = f(x)$ be the function whose graph describes the curve near the point $(1,2)$.

Then $f(1) = 2$ and $\ln([f(x)]^2 - 3) = x \cdot f(x) - 2$.

Differentiating with respect to x provides

$$\frac{2 \cdot f(x) \cdot f'(x)}{[f(x)]^2 - 3} = f(x) + x \cdot f'(x).$$

Using $f(1) = 2$, we get for $x = 1$

$$\frac{2 \cdot 2 \cdot f'(1)}{2^2 - 3} = 2 + f'(1),$$

thus $4 \cdot f'(1) = 2 + f'(1)$, so $f'(1) = \frac{2}{3}$.

Hence the equation of the tangent line is

$$y - f(1) = f'(1)(x - 1), \text{ so}$$

$$y - 2 = \frac{2}{3}(x - 1), \text{ thus}$$

$$y = \frac{2}{3}x - \frac{2}{3} + 2 = \underline{\underline{\frac{2}{3}x + \frac{4}{3}}}.$$

(b) Equation of the tangent line is $y =$ $\frac{2}{3}x + \frac{4}{3}$

(10) (a) State both parts of the Fundamental Theorem of Calculus. Use complete sentences. Be sure to include the assumptions.

(b) Find the derivative of $F(x) = \int_1^x \cos^3(t)dt$ at $x = \pi$.

(c) Determine the derivative of the function $g(x) = \int_{x^3}^1 \sin^5(t+1)dt$.

(a) Let f be a continuous function on the closed interval $[a,b]$. Then:

(I) The function $F(x) = \int_a^x f(t)dt$, $a \leq x \leq b$, is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$.

(II) If G is any antiderivative of f, then $\int_a^b f(x)dx = G(b) - G(a)$.

(b) By FTC, $F'(\pi) = \cos^3(\pi) = (-1)^3 = \underline{\underline{-1}}$.

(c) Consider the function $h(u) = \int_1^u \sin^5(t+1)dt$. By FTC, $h'(u) = \sin^5(u+1)$.

Since $g(x) = -\int_1^{x^3} \sin^5(t+1)dt = -h(x^3)$, the chain rule provides

2

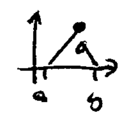
$g'(x) = -h'(x^3)\cdot 3x^2 = \underline{\underline{-3x^2\cdot\sin^5(x^3+1)}}$.

(b) $F'(\pi) =$ -1, (c) $g'(x) =$ $-3x^2\cdot\sin^5(x^3+1)$

Extra Credit Problem.

Mark the correct answers below. For each correct answer you earn 2 points, and for each incorrect answer 1 point will be subtracted. Therefore, **it might be wise to skip a question rather than risking losing a point.** However, your final score on this problem will not be negative! You need not justify your answer.

True	False	Statement
☒	☐	If a function f is increasing then it has an inverse function.
☒	☐	If f is a differentiable function such that $f(1) = -2$ and $f(3) = 4$, then there is a number c such that $f(c) = 0$. By IVT.
☐	☒	Let f be a function that is defined on a closed interval $[a, b]$. If f is not continuous then f does not have an absolute maximum value on $[a, b]$.
☐	☒	If a function f has a local maximum at a then $f'(a) = 0$.
☒	☐	If f is a continuous function that is even, then the function $F(x) = \int_1^x f(t)dt$ is odd.



f is not necessarily differentiable at a.