MA 113 — Calculus I Spring 2010
Final Exam May 5, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which of problem 8 - 10 is not to be graded by crossing through its number in the table below. Answer as many extra credit problems as you wish to; please carefully read the instructions on the last page of the exam.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Answer Key

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		12
2		8
3		9
4		12
5		10
6		9
7		10
8		15
9		15
10		15
Extra Credit		10
		100

(1) Find the first derivatives of the following functions. **Show your work!**

(a) $f(x) = 4x^2\cos(3x+1)$.

(4)

$$f'(x) = \underbrace{8x\cos(3x+1)}_{(1)} - \underbrace{4x^2}_{(1)}\underbrace{\sin(3x+1)\cdot 3}_{(2)}$$

(b) $g(x) = \dfrac{3}{\sqrt{4+6x-x^2}}$.

(4)

$$g(x) = 3\cdot(4+6x-x^2)^{-\frac{1}{2}}$$

$$g'(x) = \underbrace{-\tfrac{3}{2}}_{(1)}\underbrace{(4+6x-x^2)^{-\frac{3}{2}}}_{(1)}\cdot\underbrace{(6-2x)}_{(2)}$$

(c) $h(x) = \dfrac{e^x+2x}{x^2-3x+1}$.

(4)

$$h'(x) = \frac{\overbrace{(x^2-3x+1)(e^x+2)}^{(1)} - \overbrace{(e^x+2x)}^{(1)}\overbrace{(2x-3)}^{(1)}}{\underbrace{(x^2-3x+1)^2}_{(1)}}$$

(a) $f'(x) =$ $8x\cos(3x+1) - 12x^2\sin(3x+1)$

(b) $g'(x) =$ $-\frac{3}{2}(4+6x-x^2)^{-\frac{3}{2}}\cdot(6-2x)$

(c) $h'(x) =$ $\dfrac{(x^2-3x+1)(e^x+2)-(e^x+2x)(2x-3)}{(x^2-3x+1)^2}$

(2) Find the following limits. **Show your work!**

(a) $\lim_{x \to 3} \dfrac{\cos(3x-9) - x + 2}{x-3}$.

2

$\overset{\boxed{\frac{0}{0}}}{=} \lim_{x \to 3} \dfrac{-3\sin(3x-9) - 1}{1} \overset{1}{=} \underline{\underline{-1}}$

l'Hospital

by continuity

3

(b) $\lim_{x \to \infty} e^{-2x} \ln(x+3)$.

1

$= \lim_{x \to \infty} \dfrac{\ln(x+3)}{e^{2x}}$

2

$\overset{\boxed{\frac{\infty}{\infty}}}{=} \lim_{x \to \infty} \dfrac{\frac{1}{x+3}}{e^{2x}} \overset{1}{=} \lim_{x \to \infty} \dfrac{1}{\underbrace{\underbrace{e^{2x}}_{\to\infty} \cdot \underbrace{(x+3)}_{\to\infty}}_{\to\infty}}$

l'Hospital

1

$= \underline{\underline{0}}$

5

The reasoning $\boxed{\frac{0}{0}}$ / $\boxed{\frac{\infty}{\infty}}$ must be be given to earn 1 point at those steps.

(a) $\lim_{x \to 3} \dfrac{\cos(3x-9) - x + 2}{x-3} =$ ___ -1 ___

(b) $\lim_{x \to \infty} e^{-2x} \ln(x+3) =$ ___ 0 ___

(3) Consider the function $f(x) = x^5 - 5x$. For the following problems **be sure to justify your answer!**

(a) Find all values for x where $f(x)$ has a local maximum.

③ $f'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2-1)(x^2+1)$

$f'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

f has a local max at $x = -1$ ①

①

		-1		$+1$	
$f'(x)$	$+$		$-$		$+$
$f(x)$	↗		↘		↗

(b) Find all values for x where $f(x)$ has a local minimum.

① f has a local min at $x = 1$.

(c) Find all values for x where $f(x)$ has a point of inflection.

① $f''(x) = 20x^3 = 0$ for $x = 0$

①

		0	
$f''(x)$	$-$		$+$
$f(x)$	CD		CU

① f has an inflection point at $x = 0$

(a) $f(x)$ has a local maximum at $x = -1$

(b) $f(x)$ has a local minimum at $x = 1$

(c) $f(x)$ has a point of inflection at $x = 0$

(4) Find the following integrals.

2

(a) $\int (7x^4 + 3x^2 - 5e^x + 6\cos x)\,dx =$ $\frac{7}{5}x^5 + x^3 - 5e^x + 6\sin x + C$

3

(b) $\int_2^x \frac{3t^2+2t+4}{t}\,dt =$ $\frac{3}{2}x^2 + 2x + 4\ln|x| - 10 - 4\ln(2)$

$$= \int_2^x 3t + 2 + \frac{4}{t}\,dt = \left(\frac{3}{2}t^2 + 2t + 4\ln|t|\right)\Big|_2^x$$

$$= \frac{3}{2}x^2 + 2x + 4\ln|x| - (6 + 4 + 4\ln(2))$$

absolute value for x not necessary b/c x > 0 anyways for continuity of function between 2 and x

3

(c) $\int x\sqrt{x^2-9}\,dx =$ $\frac{1}{3}(x^2-9)^{3/2} + C$

$u = x^2 - 9$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$$\frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2-9)^{3/2} + C$$

4

(d) $\int_0^\pi t\sin(t^2)\,dt =$ $\frac{1}{2}(1 - \cos(\pi^2))$

$u = t^2$

$\frac{du}{dt} = 2t$

$\frac{1}{2}du = t\,dt$

$$\frac{1}{2}\int_0^{\pi^2} \sin(u)\,du = -\frac{1}{2}\cos(u)\Big|_0^{\pi^2}$$

$$= -\frac{1}{2}\cos(\pi^2) + \frac{1}{2}\cos(0)$$

$$= \frac{1}{2}(1 - \cos(\pi^2))$$

(5) A rock is thrown vertically upward from 30 meters above ground with a speed of 5 meters per second. Find the speed of the rock when it hits the ground. Use that the acceleration due to gravity is 10 m/sec^2 in the downward direction.

(1) $a(t) = -10$ acceleration

(1) $v(t) = -10t + C$ velocity

(1) $v(0) = C = 5$ by the given information and where we take $t = 0$ at the moment the rock is thrown.

$v(t) = -10t + 5$

(2) $h(t) = -5t^2 + 5t + C$

(1) $h(0) = C = 30$

$h(t) = -5t^2 + 5t + 30$ = height of the rock at time $t > 0$ until it hits the ground.

(2) Rock hits the ground when $h(t) = 0$.

$h(t) = -5(t^2 - t - 6) = -5(t-3)(t+2)$

Hence $h(t) = 0$ for $\boxed{t = 3 \text{ seconds}}$ as the first time after the rock has been thrown.

(2) Velocity of the rock at time $t = 3$:

$v(3) = -30 + 5 = -25$ m/sec.

Speed is: 25 m/sec.

(6) Consider the curve described by the equation

$$x^2 - xy + y^2 = 7.$$

(a) Find the derivative $\frac{dy}{dx}$. Your answer should be an expression in x and y.

4 [$2x - y - xy' + 2yy' = 0$

2 [$y' = \frac{y - 2x}{2y - x}$

(b) Find the slope of the tangent line to this curve at the point $(3, 2)$.

1 [$\text{slope} \underset{\substack{y=2\\x=3}}{=} \frac{2-6}{4-3} = -4$

(c) Find the equation of the tangent line to the curve at the point $(3, 2)$. Express it in the form $y = mx + b$.

1 [$y - 2 = -4(x - 3)$

1 [$y = -4x + 14$

(a) $\frac{dy}{dx} =$ $\frac{y-2x}{2y-x}$

(b) Slope is -4

(c) Equation is $y = -4x + 14$

(7) Consider the function $F(x) = \displaystyle\int_1^x \frac{t}{3+t^2}\,dt$.

(a) Find all interval(s) on which $F(x)$ is decreasing.

2 $F'(x) = \dfrac{x}{3+x^2}$. $F'(x) > 0$ for $x > 0$; $F'(x) < 0$ for $x < 0$. 1

1 $F(x)$ is decreasing on $(-\infty, 0)$

(b) Find all interval(s) on which $F(x)$ is concave down.

2 $F''(x) = \dfrac{(3+x^2) - x \cdot 2x}{(3+x^2)^2} = \dfrac{3-x^2}{(3+x^2)^2}$.

2 $F''(x) = 0 \iff 3 - x^2 = 0 \iff x^2 = 3$
$\iff x = \pm\sqrt{3}$

1

		$-\sqrt{3}$		$\sqrt{3}$	
$F''(x)$	$-$		$+$		$-$
$F(x)$	CD		CU		CD

1 $F(x)$ is concave down on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$

(a) $F(x)$ is decreasing on $(-\infty, 0)$

(b) $F(x)$ is concave down on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) State both parts of the Fundamental Theorem of Calculus. Use complete sentences.

1 Let f be continuous on $[a,b]$.

2 (a) Then $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$, diff^{ble} on (a,b) and $g'(x) = f(x)$.

2 (b) $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

(b) Find the derivative of $F(x) = \int_3^x \cos^4(2t+3)\,dt$.

2 $F'(x) = \cos^4(2x+3)$.

(c) Find the derivative of $G(x) = \int_{x^2}^3 \cos^4(2t+3)\,dt$.

4 $G(x) \overset{2}{=} -\int_3^{x^2} \cos^4(2t+3) \overset{2}{=} -F(x^2)$

4 $G'(x) \underset{2}{=} -F'(x^2)\cdot 2x \underset{2}{=} -2x\cdot\cos^4(2x^2+3)$

(b) $F'(x) =$ $\cos^4(2x+3)$

(c) $G'(x) =$ $-2x\cos^4(2x^2+3)$

(9) (a) State the Chain Rule. Use complete sentences and include all necessary assumptions. 9

2

If g is diff$^{\text{able}}$ at x and f is diff$^{\text{able}}$ at $g(x)$, then $f \circ g$ is diff$^{\text{able}}$ at x and we have:

1

$F := f \circ g$, then $F'(x) = f'(g(x)) \cdot g'(x)$.

2

(b) Suppose f and g are differentiable functions such that

$$f(3) = 5, \quad f'(3) = 4, \quad g(3) = 9, \quad g'(3) = 2.$$

(i) Compute $h'(3)$ where $h(x) = \ln\left(f(x)^2 + 5x\right)$.

3

$$h'(x) = \frac{2f(x) \cdot f'(x) + 5}{f(x)^2 + 5x}$$

2

$$h'(3) = \frac{2f(3) \cdot f'(3) + 5}{f(3)^2 + 5 \cdot 3} = \frac{2 \cdot 5 \cdot 4 + 5}{25 + 15}$$

$$= \frac{45}{40} = \frac{9}{8}$$

(ii) Compute $k'(3)$ where $k(x) = f(x^{-1}g(x))$.

3

$$k'(x) = f'(x^{-1}g(x)) \cdot \left[-x^{-2}g(x) + x^{-1}g'(x)\right]$$

2

$$k'(3) = f'\left(\frac{9}{3}\right) \cdot \left[-\frac{1}{9} \cdot 9 + \frac{1}{3} \cdot 2\right]$$

$$= f'(3) \cdot \left(-\frac{1}{3}\right) = -\frac{4}{3}$$

(b) $h'(3) =$ $9/8$ $\qquad k'(3) =$ $4/3$

(10) For this problem use the information that for a sphere with radius r

$$\text{the surface area is } 4\pi r^2 \text{ and the volume is } \frac{4}{3}\pi r^3.$$

A spherical balloon is being inflated so that its surface area increases at a rate of 3 cm^2/min.

(a) Find the rate at which the radius increases at the moment when the radius is 6 cm.

2 $S(t) = 4\pi r(t)^2$ 2

3 $S'(t) = 8\pi r(t) r'(t) = 3$ 1

2 $r'(t) = \dfrac{3}{8\pi r(t)}$

At time t_0 when $r(t_0) = 6$ we obtain

2 $r'(t_0) = \dfrac{3}{8\pi r(t_0)} = \dfrac{3}{8\pi \cdot 6} = \dfrac{1}{16\pi}$

(b) Find the rate at which the volume increases at the moment when the radius is 6 cm.

1 $V(t) = \dfrac{4}{3}\pi r(t)^3$

2 $V'(t) = 4\pi r(t)^2 r'(t)$

At time t_0 we obtain by (a)

$V'(t_0) = 4\pi r(t_0)^2 \cdot r'(t_0)$

3 $= 4\pi \cdot 36 \cdot \dfrac{1}{16\pi} = 9$

(a) Radius increases at a rate of $\frac{1}{16\pi}$ cm/min.

(b) Volume increases at a rate of 9 cm^3/min.

Extra Credit Problem:

Check the correct answers below. For each correct answer you earn 2 points, and for each incorrect answer 1 point will be subtracted. Therefore, it might be wise to skip a question rather than risking losing a point. However, your final score on this problem will not be negative! You need not justify your answer.

True	False	
☒	☐	If the function f is continuous on the open interval $(0, 4)$, then f has an absolute minimum on the closed interval $[1, 3]$.
☐	☒	$\int e^{4x}\,dx = \frac{1}{4x+1}e^{4x+1} + C.$
☐	☒	If $f'(a) = 0$ for some a, then the function f has a local maximum or local minimum at a.
☒	☐	$\lim_{x\to\infty} \frac{\ln(2+e^x)}{5x} = \frac{1}{5}.$
☒	☐	If $f'(a)$ exists, then $\lim_{x\to a} f(x) = f(a)$.

So l'Hospital

$$= \lim_{x\to\infty} \frac{\frac{e^x}{2+e^x}}{5} = \lim_{x\to\infty} \frac{e^x}{10+5e^x}$$

$$\boxed{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{e^x}{5e^x} = \frac{1}{5}$$