MA 113 — Calculus I Spring 2013
Exam 4 -Version 1 - Solutions April 30, 2013

Name: ______________________________

Section: ____________________________

Last 4 digits of student ID #: ________

This exam has ten multiple choice questions (five points each) and five free response questions (ten points each). Additional blank sheets are available if necessary for scratch work. No books or notes may be used. Turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities.

On the multiple choice problems:

1. You must give your *final answers* in the *multiple choice answer box* on the front page of your exam.
2. Carefully check your answers. No credit will be given for answers other than those indicated on the *multiple choice answer box*.

On the free response problems:

1. Clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. Give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each free response question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Multiple Choice Answers

Question					
1	**A**	B	C	D	E
2	A	**B**	C	D	E
3	A	B	C	**D**	E
4	**A**	B	C	D	E
5	A	**B**	C	D	E
6	**A**	B	C	D	E
7	A	B	**C**	D	E
8	A	B	C	**D**	E
9	**A**	B	C	D	E
10	A	**B**	C	D	E

Exam Scores

Question	Score	Total
MC		50
11		10
12		10
13		10
14		10
15		10
Total		100

Record the correct answer to the following problems on the front page of this exam.

1. Which of the following is an equation of the tangent line to the curve given by

$$x^2 + \sin y = xy^2 + 1$$

at the point $(1, 0)$?

A. $y = -2x + 2.$

B. $y = -x + 1.$

C. $y = 0.$

D. $y = -2x - 2.$

E. $y = 2x - 2.$

2. Let $a > 0$ be a constant. Consider the function

$$f(x) = \ln(x^2 + 2ax + a^2 + 1)$$

on the interval $(-\infty, \infty)$. Find the maximal open interval on which f is decreasing.

A. $(0, \infty).$

B. $(-\infty, -a).$

C. $(-a, \infty).$

D. $(a, \infty).$

E. The function is nowhere decreasing.

Record the correct answer to the following problems on the front page of this exam.

3. Let

$$f(x) = x \tan x + e^{2 \sin x}.$$

Find $f'(0)$.

A. e

B. -1

C. $\pi/2$

D. 2

E. -2

4. Evaluate the following limit

$$\lim_{x \to 0} \frac{\ln(1+x) - x}{x^2}.$$

A. $-\frac{1}{2}$

B. $\frac{1}{2}$

C. 0

D. $\ln 2$

E. The limit does not exist.

Record the correct answer to the following problems on the front page of this exam.

5. Find the linearization of $f(x) = \dfrac{x}{1+x^2}$ at $x = 2$.

A. $L(x) = \frac{2}{5}$

B. $L(x) = -\frac{3}{25}x + \frac{16}{25}$

C. $L(x) = \frac{3}{25}x - \frac{1}{25}$

D. $L(x) = x + 2$

E. $L(x) = \frac{2}{5}x$

6. At a given moment, a plane passes directly above a radar station at an altitude of 6 km. The plane's speed is 480 km/h. How fast is the distance between the plane and the station changing one minute later?

A. 384 km/h

B. $240\sqrt{2}$ km/h

C. $240\sqrt{3}$ km/h

D. $480\sqrt{2}$ km/h.

E. $480\sqrt{3}$ km/h

Record the correct answer to the following problems on the front page of this exam.

7. Which of the following statements is NOT true?

A. If $f(x)$ is continuous on $[a, b]$, then f attains an absolute minimum on the interval.

B. If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for some constant C.

C. If $\int_0^1 f(x)\,dx \geq 0$, then $f(x) \geq 0$ for all $x \in [0, 1]$.

D. If f is continuous on $[0, 1]$ and $f(0) \neq f(1)$, then there exists $c \in (0, 1)$ such that

$$f(c) = \frac{1}{2}(f(0) + f(1)).$$

E. If f is differentiable at a, then $\lim_{x \to a} f(x) = f(a)$.

8. Suppose that $f'(x) = \frac{1}{x^2+4}$ for all $x > 0$ and $f(2) = 0$. Find $f(2\sqrt{3})$.

A. $\frac{\pi}{2}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{24}$

E. 1

Record the correct answer to the following problems on the front page of this exam.

9. A particle is traveling along a straight line with a velocity of

$$v(t) = t^2 - 7t + 6 \text{ meters/minute.}$$

What is the particle's <u>total</u> distance traveled during the time interval $[0, 4]$?

A. $\frac{49}{3}$ meters

B. $\frac{91}{6}$ meters

C. 15 meters

D. 16 meters

E. $\frac{32}{3}$ meters

10. Determine the integral

$$\int \frac{x^2\sqrt{x} + 2}{x}\, dx.$$

A. $\frac{2}{7}x^3\sqrt{x} + 2x + C.$

B. $\frac{2}{5}x^2\sqrt{x} + 2\ln x + C.$

C. $\frac{2}{7}x^3 + 2x - \ln x + C.$

D. $2x^2\sqrt{x} + \ln x + C.$

E. $3x^2\sqrt{x} - 2x + C.$

11. A rectangular box of height h with square base of length b has volume $V = 4$ cubic meters. Two of the side faces are made of material costing \$40 per square meter. The remaining sides, and the top and bottom, cost \$20 per square meter. Which values of b and h minimize the cost of the box?

Solution: Let C denote the cost of the box. Then

$$C = (bh + bh) \cdot 40 + (bh + bh + b^2 + b^2) \cdot 20 = 80bh + 40(bh + b^2).$$

Since

$$V = b^2 h = 4,$$

we obtain $h = \frac{4}{b^2}$. Hence,

$$C = C(b) = 80 \cdot b \cdot \frac{4}{b^2} + 40(b \cdot \frac{4}{b^2} + b^2) = 40b^2 + \frac{480}{b},$$

where $0 < b < \infty$. To find the absolute minimum value of $C(b)$ on the interval $(0, \infty)$, we take the derivative

$$C'(b) = 80b - \frac{480}{b^2}.$$

Set $C'(b) = 0$, we obtain

$$80b = \frac{480}{b^2}; \text{ i.e. } b^3 = 6.$$

Thus, $b = \sqrt[3]{6}$ is the critical number for the function $C(b)$. Since $C'(b) < 0$ on $(0, \sqrt[3]{6})$ and $C'(b) > 0$ on $(\sqrt[3]{6}, \infty)$, we know that $C(b)$ is decreasing on $(0, \sqrt[3]{6})$ and increasing on $(\sqrt[3]{6}, \infty)$. It follows that $C(b)$ has the absolute minimum at $b = \sqrt[3]{6}$. Thus the values of b and h that minimize the cost of the box are

$$b = \sqrt[3]{6} \approx 1.8171 \quad \text{and} \quad h = \frac{4}{b^2} = \frac{4}{\sqrt[3]{36}} \approx 1.2114.$$

Grading Guidelines:
1. 3 pts for the correct formula of the cost function.
2. 3 pts for the correct critical number.
3. 1 pt. for displaying the correct values of b and h.
4. 3 pts for justifying that the function has the absolute minimum value at the critical number.

12. (a) How long will it take for \$4,000 to double in value if it is deposited in an account earning 7% interest, compounded continuously? Recall that the value of the account after t years is $P(t) = P_0 e^{rt}$.

Solution: Since $P(t) = P_0 e^{rt} = 4000e^{0.07t}$, the doubling time is

$$T = \frac{\ln 2}{r} = \frac{\ln 2}{0.07} \approx 9.9.$$

It will take about 9.9 years for \$4,000 to double in value with the interest rate of 7%.

Grading Guidelines:
5 pts. for the correct answer

(b) After winning \$5 million in the state lottery, Jessica learns that she will receive five yearly payments of \$1 million beginning immediately. What is the PV of Jessica's prize, if $r = 6\%$? Recall that the present value (PV) of P dollars received at time t is Pe^{-rt}.

Solution: The total PV of the prize is the sum of the PV of the \$1 million received at year 0, the PV of the \$1 million received at year 1, the PV of the \$1 million received at year 2, the PV of the \$ 1 million received at year 3, and the PV of the \$1 million received at year 4. Thus,

$$\begin{aligned}\text{The PV of the prize} &= 1 + e^{-0.06\times 1}\cdot 1 + e^{-0.06\times 2}\cdot 1 + e^{-0.06\times 3}\cdot 1 + e^{-0.06\times 4}\times 1\\ &= 1 + e^{-0.06} + e^{-0.12} + e^{-0.18} + e^{-0.24}\\ &\approx 4.45058304.\end{aligned}$$

The PV of Jessica's prize is \$4,450,583.04.

Grading Guidelines:
5 pts. for the correct formula and correct answer.
2 pts. for the incorrect formula $e^{-0.06} + e^{-0.12} + e^{-0.18} + e^{-0.24} + e^{-0.30}$**.**
2 pts. for using the continuous model.

Free Response Questions: Show your work!

13. Evaluate the following integrals.

(a) (5pts.)

$$\int_0^{\frac{1}{2}} \left(12y^2 + 6y\right)\, dy$$
$$= \left\{4y^3 + 3y^2\right\} \Big|_0^{\frac{1}{2}}$$
$$= \frac{5}{4}.$$

3 pts. for displaying the correct antiderivative.
2 pts. for the correct answer.

(b) (5pts.)

$$\int_{-1}^{2} \sqrt{5x+6}\, dx$$

Let $u = 5x + 6$. Then $du = 5\, dx$. Hence, $dx = \frac{1}{5}\, du$. It follows that

$$\int_{-1}^{2} \sqrt{5x+6}\, dx = \int_1^{\sqrt{16}} \sqrt{u} \cdot \frac{1}{5}\, du$$
$$= \frac{1}{5} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{16}$$
$$= \frac{2}{15}\,(64 - 1)$$
$$= \frac{42}{5} = 8.4.$$

3 pts. for displaying the correct antiderivative.
2 pts. for the correct answer.

14. (a) Suppose that f is continuous on the interval $[0, 2]$ and

$$\int_0^x f(t)\,dt = x^2 + x \qquad \text{for } 0 \le x \le 2.$$

Find $f(1)$.

Solution (5pts.): By the Fundamental Theorem of Calculus,

$$f(x) = \frac{d}{dx}\left\{\int_0^x f(t)\,dt\right\} = \frac{d}{dx}\{x^2 + x\} = 2x + 1.$$

Hence,

$$f(1) = 2 \cdot 1 + 1 = 3.$$

(b) Find the derivative of the function g defined by

$$g(x) = \int_2^{\sqrt{x}} \frac{\cos(t^2)}{1+t}\,dt.$$

Solution (5pts.): By the Fundamental Theorem of Calculus,

$$\begin{aligned} g'(x) &= \frac{\cos((\sqrt{x})^2)}{1+\sqrt{x}} \cdot \frac{d}{dx}\{\sqrt{x}\} \\ &= \frac{\cos x}{1+\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\cos x}{2(\sqrt{x}+x)}. \end{aligned}$$

15. Compute the area of the region enclosed by the graphs of the two functions

$$f(x) = 20 + x - x^2 \quad \text{and} \quad g(x) = x^2 - 5x.$$

Present also a sketch of the two graphs.

Solution: To find the coordinates of the intersections, we set $f(x) = g(x)$. This gives

$$20 + x - x^2 = x^2 - 5x.$$

Hence, $x^2 - 3x - 10 = 0$. Since $x^2 - 3x - 10 = (x - 5)(x + 2)$, we obtain $x = -2$ and $x = 5$. Thus,

$$\begin{aligned} \text{Area} &= \int_{-2}^{5} \{f(x) - g(x)\}\, dx \\ &= \int_{-2}^{5} \{20 + 6x - 2x^2\}\, dx \\ &= \{20x + 3x^2 - \frac{2}{3}x^3\}_{-2}^{5} \\ &= \frac{343}{3} = 114\frac{1}{3}. \end{aligned}$$

Grading Guidelines:
1. 2 pts. for the graphs.
2. 2 pts. for the x coordinates of the intersections.
3. 3 pts. for setting up the integral.
4. 3 pts. for evaluating the integral.