Worksheet # 18: Extreme Values and the Mean Value Theorem

1. (a) Define the following terms or concepts:
   - Critical point
   - f has a local maximum at $x = a$
   - Absolute maximum

   (b) State the following:
   - The First Derivative Test for Critical Points
   - The Mean Value Theorem

2. Sketch the following:

   (a) The graph of a function defined on $(-\infty, \infty)$ with three local maxima, two local minima, and no absolute minima.

   (b) The graph of a continuous function with a local maximum at $x = 1$ but which is not differentiable at $x = 1$.

   (c) The graph of a function on $[-1, 1)$ which has a local maximum but not an absolute maximum.

   (d) The graph of a function on $[-1, 1]$ which has a local maximum but not an absolute maximum.

   (e) The graph of a discontinuous function defined on $[-1, 1]$ which has both an absolute minimum and absolute maximum.

3. Find the critical points for the following functions:

   (a) $f(x) = x^4 + x^3 + 1$

   (b) $g(x) = e^{3x}(x^2 - 7)$

   (c) $h(x) = |5x - 1|$

4. Find the absolute maximum and absolute minimum values of the following functions on the given intervals. Specify the x-values where these extrema occur.

   (a) $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

   (b) $h(x) = x + \sqrt{1 - x^2}$, $[-1, 1]$

5. (a) Consider the function $f(x) = 2x^3 - 9x^2 - 24x + 5$ on $(-\infty, \infty)$.

   i. Find the critical point(s) of $f(x)$.

   ii. Find the intervals on which $f(x)$ is increasing or decreasing.

   iii. Find the local extrema of $f(x)$.

   (b) Repeat with the function $f(x) = \dfrac{x}{x^2 + 4}$ on $(-\infty, \infty)$.

   (c) Repeat with the function $f(x) = \sin^2(x) - \cos(x)$ on $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$.

6. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   (a) $f(x) = \dfrac{x}{x + 2}$ on the interval $[1, 4]$

   (b) $f(x) = \sin(x) - \cos(x)$ on the interval $[0, 2\pi]$

7. Comprehension check:

   (a) True or False: If $f'(c) = 0$ then f has a local maximum or local minimum at c.

   (b) True or False: If f is differentiable and has a local maximum or minimum at $x = c$ then $f'(c) = 0$.

   (c) A function continuous on an open interval may not have an absolute minimum or absolute maximum on that interval. Give an example of continuous function on $(0, 1)$ which has no absolute maximum.

   (d) True or False: If f is differentiable on the open interval (a, b), continuous on the closed interval $[a, b]$, and $f'(x) \neq 0$ for all x in (a, b), then we have $f(a) \neq f(b)$.