Record the correct answer to the following problems on the front page of the exam.

We discuss solutions to an old exam 3 (F-12) and also offer helpful comments for preparation for the upcoming exam 3 in Ma 113.

First some general suggestions:

- Remember the formula for linear approximation and its various uses, including the meaning of differentials.
- Review the limits at infinity as well as infinite limits. Be sure to learn the mechanism of L'Hopital's rule and when not to use it! Do not mess up your quotient rule for derivatives with it!
- Be sure that you can precisely state MVT and the theorem on the existence of absolute extremes for continuous functions on closed intervals.
- Do not forget other important theorems: These include IVT, Fermat's Theorem, Rolle's Theorem etc.
- Review what is meant by an increasing and decreasing function and how to test for it.
- Be sure to review the exact meaning and conditions for extremum points, absolute and local max/min as well, as the first and second derivative tests for these.
- Be sure to make a table of anti-derivatives for your memory. Be sure not to confuse with the derivative formulas.
- Learn to handle sums using the summation notation.
- Remember the needed formulas for Newton's method, estimation of definite integrals as well as their calculation by a limit.

1. The function f satisfies $f'(x) < 0$ for all x and $f''(x) < 0$ for all x. Which of the following could be the graph of f?

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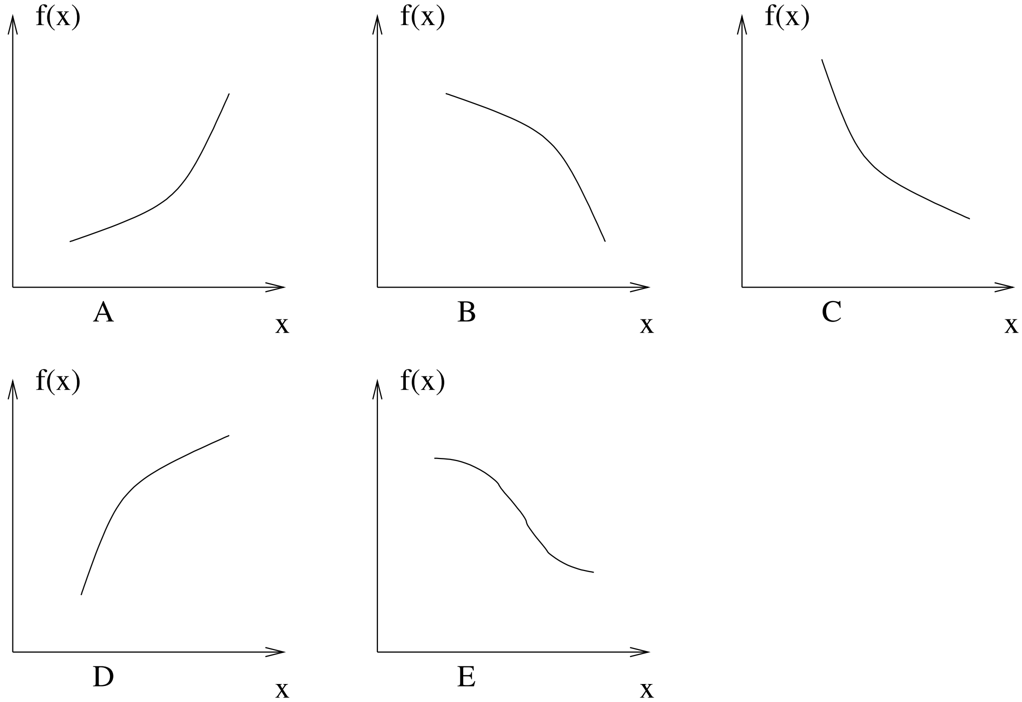

Option B. We note that $f'(x) < 0$ tells us that we look for a decreasing function, so discard A and D. Now $f''(x) < 0$ says it is always concave down. Note that C is concave up and E has mixed concavity. Thus, B is the only choice satisfying all conditions.

2. Let the function f be defined by $f(x) = |\sin(x)|$. How many critical points does the function f have on the open interval $(\pi/4, 5\pi/4)$?

A. 0

B. 1

C. 2

D. 3

E. 4

The function $|\sin(x)|$ is equal to $\sin(x)$ on $(\pi/4, \pi]$ and $-\sin(x)$ on $(\pi, 5\pi/4)$. The derivative is then $cos(x)$ on the first interval and $-cos(x)$ on the second. The derivative is zero at $\pi/2$ and at π it is undefined by the following calculation.
The limit $\lim_{x\to\pi^-} \frac{|\sin(x)|-|\sin(\pi)|}{x-\pi} = \cos(\pi) = -1$ as evident from the first interval formula. The corresponding limit $\lim_{x\to\pi^+} \frac{|\sin(x)|-|\sin(\pi)|}{x-\pi} = -\cos(\pi) = 1$ as evident from the second interval formula. Thus, we have two critical points.
So, it is option C.

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3. Let f be a function whose domain is the interval $(0, 2)$ and assume that f is differentiable on $(0, 2)$. Select the statement that must be true for any such f.

A. If f has a local maximum at 1, then $f'(1) = 0$.

B. If $f'(1) = 0$, then f has a local maximum at 1.

C. If $f(1) = 2012$, then f has a local maximum at 1.

D. If $f''(1) > 0$, then f has a local minimum at 1.

E. If $f(1/2) = f(3/2)$, then $f'(1) = 0$.

Option A is definitely true since it follows from Fermat's theorem. If we just know that $f'(1) = 0$, then we only know that it is a critical point. Further analysis is needed to decide its nature. So, B fails. Option C has no chance to be true, without knowing the nature of the function. The second derivative positive says that the function is concave up, but without the first derivative being zero, it does not give an extremum. So D fails. E fails because it is even more spurious that C!

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4. Suppose that f is a function whose domain is the open interval $(-1, 1)$ and f has a local maximum at 0. One of the statements below can never be true for f. The other four will be true for some choices of f and false for other choices. Which of the following can never be true for any such f?

A. $f'(0) = 0$ and $f''(0) < 0$

B. $f'(0) = 0$ and $f''(0) = 0$

C. $f'(0)$ does not exist

D. $f'(0) = 0$ and $f'(x) < 0$ for $0 < x < 1$

E. $f'(0) = 0$ and $f'(x) > 0$ for $0 < x < 1$

We know that the condition for the local maximum says that $f'(0) = 0$ or $f'(0)$ does not exist and in addition $f(x) \leq f(0)$ in some interval containing 0. We try to make an example of local maximum for each choice and see which fails. For A, we can take $f(x) = -x^2$. For B, take $f(x) = -x^4$. For C, take $f(x) = -|x|$. For D, take the same example as A.
Thus, E is the candidate! The conditions actually imply that for any $x \in (0, 1)$ we get $f(x) - f(0) = f'(c)(x - 0) > 0$ for some $c \in (0, x)$, by the MVT. This is contradictory to 0 being a local maximum!

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5. Suppose that f is a function on the open interval $(0, 3)$ and we know the following information about the derivative f':

$$\begin{array}{ll} f'(x) > 0, & 0 < x < 1 \\ f'(1) = 0, & \\ f'(x) < 0, & 1 < x < 2 \\ f'(2) = 0, & \\ f'(x) < 0, & 2 < x < 3 \end{array}$$

Which of the following is true?

A. The function f has a local maximum at 1 and a local minimum at 2.

B. The function f has a local maximum at 1 and a local maximum at 2.

C. The function f has a local minimum at 1 and a local minimum at 2.

D. The function f has a local minimum at 1 and a local maximum at 2.

E. The function f has a local maximum at 1 and does not have an extremum at 2.

Note that $x = 1$ is a c.p. with f' changing from positive to negative. So it is a local max. The point $x = 2$ is a c.p. with f' having the same negative sign on either side. So it is not an extremum point! Thus, E is the answer.

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6. We let $f(x) = x^2 - 8$ and use Newton's method to find a solution of $f(x) = 0$. If $x_1 = 4$, find the exact value of x_3.

A. $\sqrt{8}$

B. 8

C. 3

D. 17/6

E. None of the above.

Note that $f(x)/f'(x) = (x^2 - 8)/(2x) = x/2 - 4/x$. So, our iterative formula is $x \to x - (x/2 - 4/x) = x/2 + 4/x$. We start with $x_1 = 4$, then $x_2 = 4/2 + 4/4 = 3$. Then $x_3 = 3/2 + 4/3 = 17/6$. So, option D.

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7. Let $(a, f(a))$ be the point where the tangent line to the graph of $f(x) = x\cos(x)$ is horizontal. We use Newton's method to find $x_1, x_2, x_3, \ldots,$ the successive approximations to a. Give the formula that we use to compute x_{n+1} from x_n.

A. $x_{n+1} = x_n - \dfrac{x_n \cos(x_n)}{\cos(x_n) - x_n \sin(x_n)}$

B. $x_{n+1} = x_n - \dfrac{\cos(x_n) - x_n \sin(x_n)}{x_n \cos(x_n)}$

C. $x_{n+1} = x_n + \dfrac{\cos(x_n)}{\sin(x_n)}$

D. $x_{n+1} = x_n + \dfrac{2\sin(x_n) + x_n \cos(x_n)}{\cos(x_n) - x_n \sin(x_n)}$

E. $x_{n+1} = x_n + \dfrac{\cos(x_n) - x_n \sin(x_n)}{2\sin(x_n) + x_n \cos(x_n)}$

To solve $f'(x) = 0$, use the iteration $x \to x - f'(x)/f''(x)$. We have $f(x) = x\cos(x)$, $f'(x) = \cos(x) - x\sin(x)$, and $f''(x) = -2\sin(x) - x\cos(x)$. So, the iteration is $x \to x - \dfrac{\cos(x) - x\sin(x)}{-2\sin(x) - x\cos(x)}$. This simplifies to give option E.

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8. Which of the following is **not** an anti-derivative of $2\sin(x)\cos(x)$?

A. $1 + \sin^2(x)$

B. $\sin^2(x)$

C. $1 - \cos^2(x)$

D. $-\cos^2(x)$

E. $\sin^2(x) + \cos^2(x)$

Option E. The derivative of $1 = \sin^2(x) + \cos^2(x)$ is 0 and $\sin(x)\cos(x)$ is not zero. Thus the function in E is not an antiderivative of $\sin(x)\cos(x)$. We could also quickly check the derivatives of the first four functions to be $2\sin(x)\cos(x)$, or note that they differ from each other by a constant (but this requires some trigonometric skill!) You could also note that the function in E does not differ from the other functions by a constant.

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9. We have a sequence of numbers $\{a_1, a_2, a_3, \ldots\}$ so that $\sum_{k=1}^{n} a_k = n^2$ holds for $n = 1, 2, 3, \ldots$.

Find the value of $\sum_{k=1}^{12} (4a_k + 2)$.

A. 578

B. 600

C. 2306

D. 2328

E. 2400

Option B. $\sum_{k=1}^{12}(4a_k + 2) = 4\sum_{k=1}^{12} a_k + \sum_{k=1}^{12} 2 = 4 \cdot 12^2 + 24 = 600$.

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10. Let $f(x) = 3-x$. Subdivide the interval $[1, 3]$ into four equal sub-intervals and compute R_4, the value of the right-endpoint approximation to the area under the graph of f on the interval $[1, 3]$.

A. 5/2

B. 2

C. 3/2

D. 1

E. None of the above.

Note that the length of each interval is $(3 - 1)/4 = 1/2$. First identify all division points as $[1, 1+1/2, 1+2/2, 1+3/2, 1+4/2] = [1, 3/2, 2, 5/2, 3]$. The function values are $[2, 3/2, 1, 1/2, 0]$. Thus the estimate R_4 is $(1/2)(3/2 + 1 + 1/2 + 0) = 3/2$ since we want the right sum. For left sum, we take the first 4 values of the function instead.

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11. The volume of a sphere of radius r is $V(r) = 4\pi r^3/3$.

(a) Find $L(r)$, the linearization of the volume V at $r = 10$.

(b) A sphere has radius of 10 centimeters. The sphere is heated and the radius increases 6%. Use the linearization to approximate the increase in the volume of the sphere. Please give your answer as a multiple of π.

Free Response Questions: Show your work!

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a) Since $V(r) = (4/3)\pi r^3$, $V'(3) = 4\pi r^2$. The linearization at $r = 10$ is

$$L(r) = V'(10)(r - 10) + V(10) = 400\pi(r - 10) + \frac{4000\pi}{3}.$$

b) The increase in the radius is 6% or 0.6 centimeters. The resulting new volume is estimated to be $L(10.6) = 400\pi(0.6) + \frac{4000\pi}{3}$. So, the increase is $400\pi(0.6) + \frac{4000\pi}{3} - \frac{4000\pi}{3} = 400\pi(0.6) = 240\pi$.
It is also possible to deduce the same answer by differential approximation which gives $dV = V'(10)dr = 400\pi(0.6)$.

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12. Let $f(x) = x^3 + 3x^2 - 9x + 2012$.

(a) Find the open intervals where f is increasing or decreasing.

(b) Find the intervals where f is concave up or concave down.

(c) Give all inflection points of f.

Use calculus to justify your answers.

We do all necessary calculations first and then answer the questions.
First we calculate the derivatives:

First derivative: $$f'(x) = 3x^2 + 6x - 9 = 3(x + 3)(x - 1)$$

Second Dericvative: $$f''(x) = 6x + 6 = 6(x + 1)$$

Thus the points of interest are $x = -3, x = -1, x = 1$. This gives four intervals

$$I_1 = (-\infty, -3), I_2 = (-3, -1), I_3 = (-1, 1), I_4 = (1, \infty).$$

We record the behavior of all functions in a tabular form. The signs are checked by test points in each interval.

x	I_1	I_2	I_3	I_4
$f'(x)$	positive	negative	negative	positive
Behavior of f	increasing	decreasing	decreasing	increasing
$f''(x)$	negative	negative	positive	positive
Concavity status	down	down	up	up

The answers are now easy to give.
(a) Increasing: I_1, I_4 i.e. $(-\infty, -3)$ and $(1, \infty)$ since $f'(x) > 0$.
Decreasing: The remaining interval $(-3, 1)$ since $f'(x) < 0$.
(b) Concave up: $(-\infty, -1)$ since $f''(x) > 0$. Concave down: $(-1, \infty)$ since $f''(x) < 0$.
(c) Inflection point: $x = -1$, where the concavity changes.

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13. Evaluate the following limits. Explain your reasoning.

(a) $\lim_{x \to 1} \frac{\ln(x)}{x^2 - 1}$.

This limit is in indeterminate form 0/0 and we use L'Hôpital's rule to evaluate the limit. $\lim_{x\to 1} \frac{\ln(x)}{x^2-1} = \lim_{x\to 1} \frac{1/x}{2x} = \frac{1/1}{2\cdot 1} = 1/2$. The last limit is evaluated by the basic quotient rule of limits.

(b) $\lim_{x \to 1} \frac{e^x}{x^2 - 2}$

Separate limits of numerator and denominator come out e and -1, since both these functions are continuous. Since the ratio is well defined, the limit of the resulting function is simply $\frac{e}{-1} = -e$ by the quotient rule of limits.

(c) $\lim_{x \to 0} \frac{1 - \cos(3x)}{x^2}$

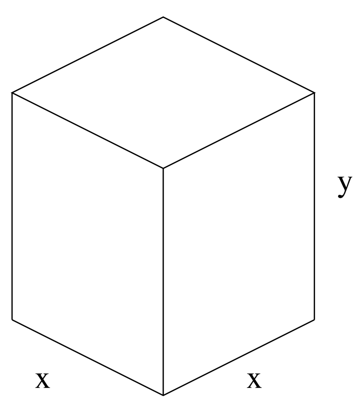
This is also an indeterminate $\frac{0}{0}$ form and so we apply L'Hôpital's rule to evaluate the limit.
Thus: $\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \lim_{x \to 0} \frac{3\sin(3x)}{2x}$. At this stage, we could rewrite the function as $\frac{3}{2/3} \frac{\sin(3x)}{3x}$ and recall the known result that $\sin(\theta)/\theta$ has limit 1, if θ tends to 0. This will yield the answer $\frac{3}{2/3} = 9/2$.
We could also repeat the L'Hôpital's rule and get that the desired limit is equal to $\lim_{x \to 0} \frac{9\cos(3x)}{2}$. Now the limit is evaluated by the quotient rule again to be $9/2$.

14. A large shipping box has a square base of sidelength x meters. The height of the container is y meters. Central Associated Transport Services (CATS) will only accept the box if the sum of the height and the perimeter of the base is equal to 10 meters.

(a) Write down a function which gives the volume of the box in terms of x, the sidelength of the base. Give the domain of the function which gives the volume.

(b) Find the dimensions x and y of the box with the **largest** possible volume.

(c) Explain how we know we have found the largest possible volume.



a) The volume of the box is $V = x^2 y$. We have the restriction that $4x + y = 10$.
Such a problem can be solved by using any convenient variable as the main variable, however, since the question specifically asks to write the volume in terms of x, we eliminate y.
The condition gives $y = 10 - 4x$ and hence $V(x) = x^2(10 - 4x)$.
Next, we determine the domain. For physical reasons, we must have $x \geq 0$ and $y = 10 - 4x \geq 0$

so that $0 \leq x \leq 10/4 = 2.5$. Hence the domain is $[0, 2.5]$.
b) To find the maximum of $V(x)$, we need to list the critical points and the endpoints.
We compute $V'(x) = 2x(10-4x) + x^2(-4) = 2x(10-4x-2x) = 2x(10-6x)$. $V'(x)$ always exists and is zero at $x = 0, x = 10/6 = 5/3$.
Technically, for our book, the only critical point is $x = 5/3$ since the other is an endpoint.
The endpoints are $x = 0, x = 2.5$. Now, we get:

x	0	$5/3$	2.5
$V(x)$	0	$(5/3)^2(10-4(5/3))$	0

The largest volume is $(5/3)^2(10-4(5/3)) = 250/27$ cubic meters (or approximately 9.3 cubic meters.) The volume occurs for a box with $x = 5/3$ meters and $y = 10/3$ meters.
c) We know that a continuous function on a closed interval has an absolute maximum largest and it must occur at a critical point or endpoint.
Instead of using the theorem, we could argue that the largest value occurs when $x = 5/3$ since $V(x)$ is increasing on $[0, 5/3]$ and decreasing on $[5/3, 5/2]$.
We could also deduce that $x = 5/3$ gives a local maximum since $V''(5/3) < 0$ and hence $x = 5/3$. However, this is not enough to claim absolute maximum, without further analysis.

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15. Let g be a twice differentiable function and suppose that $g''(x) = 3x + \cos(x)$. If $g'(0) = 2$ and $g(1) = 3$, find the function g.

Using anti-derivative of g'' we deduce that $g'(x) = 3x^2/2 + \sin(x) + A$ for some constant A.
Using $g'(0) = 2$ we get $2 = 3/2(0) + \sin(0) + A$ so that $A = -2$.
Taking anti-derivative of g' we deduce that $g(x) = (3/2)x^3/3 - \cos(x) + 2x + B = (1/2)x^3 - \cos(x) + 2x + B$.
Using the condition $g(1) = 3$ we deduce that $3 = (1/2) - \cos(1) + 2 + B$ so $B = 1/2 + \cos(1)$.
Thus

$$g(x) = \frac{x^3}{2} - \cos(x) + 2x + \frac{1}{2} + \cos(1).$$
