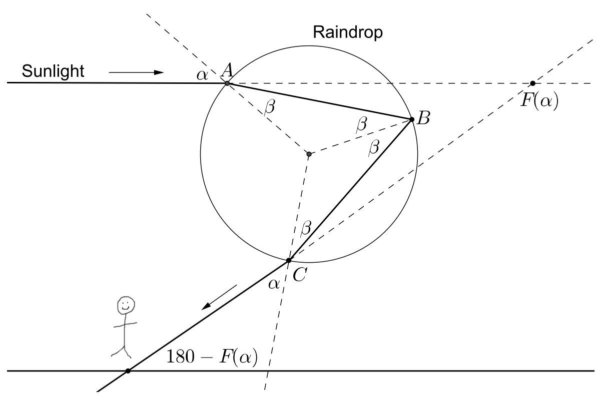
MA 113 Calculus I, Fall 2013
Written Assignment #6
Due Wednesday, November 6, 2013, at beginning of lecture

Instructions: The purpose of this assignment is to develop your ability to formulate and communicate mathematical arguments. Your complete assignment should have your name and section number on each page, be stapled, and be neat and legible. *Unreadable work will receive no credit.*

You should provide well-written, complete answers to each of the questions. We will look for correct mathematical arguments, complete explanations, and correct use of English. Your solution should be formulated in complete sentences. As appropriate, you may want to include diagrams or equations written out on a separate line. You may read your textbook to find examples of how we communicate mathematics.

Students are encouraged to use word-processing software to produce high quality solutions. However, you may find that it is simpler to add graphs and equations using pen or pencil.

In this worksheet we will learn how the appearance of rainbows can be explained using calculus.



The solid lines in the figure show the path of a ray of sunlight as it passes through a water droplet. The light ray changes direction or is refracted as it passes from air to water at points A and C, and it is reflected at point B. The angle of deviation, $F(\alpha)$, is given by

$$F(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta,$$

where α is the angle of incidence of the sunlight when entering the raindrop and β is the angle of refraction. Snell's law tells us that the relation between these angles is

$$\sin\alpha = k \sin\beta,$$

where the index of refraction k is a constant that depends on the medium through which the light passes.

Suppose we know the angle(s) α_0 for which $F'(\alpha_0) = 0$. Then this tells us that the rate of change of $F(\alpha)$ is very small for α close to α_0. In other words, light entering the raindrop at approximately the angle α_0 is turned by almost the same amount. For an observer this leads to a concentration of refracted light at the angle $180° - F(\alpha_0)$, and this leads to the appearance of the rainbow for the observer.

The goal is to find the value(s) of α for which $F'(\alpha) = 0$.

1. (7 Points) The position of the rainbow

   (a) Find the value of $\frac{d\beta}{d\alpha}$ for which $\frac{dF}{d\alpha} = 0$.

   (b) Consider Snell's law $\sin\alpha = k\sin\beta$. Use differentiation and your result from (a) to find an identity for $\cos\alpha$ and $\cos\beta$.

   (c) Use Snell's law, your result from (b), and some trigonometric identities to derive

   $$\sin^2\beta = \frac{4-k^2}{3k^2} \text{ for the values of } \alpha \text{ at which } \frac{dF}{d\alpha} = 0.$$

   (d) For the medium water, as in raindrops, the constant k is approximately $\frac{4}{3}$. Use this value and your calculator to find approximate values for β, α, and $F(\alpha)$. Your have to show your work and make clear at which step you use the calculator.
   Give your final answers also in degrees, and compute the value of $180 - F(\alpha)$, which is the angle at which the observer sees the rainbow.

2. (3 Points) The colors of the rainbow

   The precise value of k depends on the wavelength of the light. For red light, $k \approx 1.3318$, while for violet light $k \approx 1.3435$. Find out whether red appears above or below violet in the rainbow, and use Problem 1. (c) to explain your answer.

3. (1 Point Extra Credit) Why can't you go to the end of the rainbow (... and find the pot of gold)?