MA 113 Calculus I, Spring 2017 Written Assignment #6

Due Friday, March 31, 2017, at the beginning of lecture

Instructions: The purpose of this assignment is to develop your ability to formulate and communicate mathematical arguments. Your complete assignment should have your name and section number on each page, be stapled, and be neat and legible. *Unreadable work will receive no credit.*

You should provide well-written, complete answers to each of the questions. We will look for correct mathematical arguments, complete explanations, and correct use of English. Your solution should be formulated in complete sentences. As appropriate, you may want to include diagrams or equations written out on a separate line. You may read your textbook to find examples of how we communicate mathematics.

Students are encouraged to use word-processing software to produce high quality solutions. However, you may find that it is simpler to add graphs and equations using pen or pencil.

An important application of calculus is the approximation of square, cube, and other roots of numbers. In this assignment, we will use *Newton's Method* to approximate roots. To learn more about Newton's method, read section 4.8 in your textbook.

The goal in this problem is approximate $\sqrt[n]{m}$ where $n \geq 2$ and $m \geq 1$ are positive integers. The first step in Newton's method is to identify an equation for which $\sqrt[n]{m}$ is a solution. Observe that $\sqrt[n]{m}$ is a solution of the equation $x^n - m = 0$.

1. (1 point) The second step in Newton's method is to select the smallest n-th power of integers that is greater than m, which we write as $(x_0)^n$. For example, if $n = 2$, then the second powers of integers are $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, etc. If $n = 3$ and $m = 11$, what is x_0 in this case? (Note: You MUST get this answer correct in order to receive credit on the remaining problems. If you miss this, you will not receive partial credit for the remainder of the work.)

2. (2 points) The value x_0 serves as an initial approximation of $\sqrt[n]{m}$. The third step in Newton's method is to improve this approximation using calculus. For any $c > 0$, use techniques from calculus to show that the tangent line to $f(x) = x^n - m$ at the point $(c, f(c))$ lies below the graph of f on the interval $(0, \infty)$.

3. (2 points) Show that the tangent line to $f(x)$ at the point $(x_0, f(x_0))$ intersects the x-axis at the x-coordinate
$$x_0 - \frac{f(x_0)}{f'(x_0)}\,.$$
Using your answer to the previous problem, explain why this point is greater than $\sqrt[n]{m}$ but less than x_0.

4. (1 point) Write
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}\,.$$
Find the value of x_1 when $n = 3$ and $m = 11$.

5. (2 points) Consider again the general case where $n \geq 2$ and $m \geq 1$. Show that the tangent line to $f(x)$ at the point $(x_1, f(x_1))$ intersects the x-axis at the x-coordinate

$$x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Explain why this point is greater than $\sqrt[n]{m}$ but less than x_1.

6. (1 point) Write

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Find the value of x_2 when $n = 3$ and $m = 11$.

7. (1 point) Newton's method works by continuing this approximation process repeatedly. For $n = 3$ and $m = 11$, find x_3, x_4, and x_5 where $x_j = x_{j-1} - \frac{f(x_{j-1})}{f'(x_{j-1})}$. **Show your work.** What is the difference in the first five decimal places between this approximation and the value of $\sqrt[3]{11}$ given by wolframalpha or your calculator?