Higher-Order Approximations Using Taylor Polynomials
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1. Estimating $\sqrt{2}$

We will estimate $\sqrt{2}$ by finding a polynomial that approximates the function $\sqrt{1+x}$, and then evaluate that polynomial at $x = 1$ to estimate $\sqrt{1+1} = \sqrt{2}$.

Step 1: Linear Approximation. If we use the linearization of $f(x) = \sqrt{1+x}$ at the point $a = 0$, then we have

$$L(x) = f'(0)(x-0) + f(0) = \frac{1}{2}x + 1\,.$$

Thus, we have $\sqrt{1+1} \approx L(1) = 3/2 = 1.5$. Since the true value of $\sqrt{2}$ is $1.41421\ldots$, this is a reasonable but not particularly accurate approximation of $\sqrt{2}$. We need something better!

Step 2: Quadratic Approximation. The key idea to higher-order approximations is to realize that the higher derivatives of $f(x)$ have a role to play. The linearization of $f(x)$, which is frequently called a "first-order approximation", only involves the first derivative. Can we use the second derivative somehow? Yes! Using $f(x) = \sqrt{1+x}$, define the second-order approximation of $f(x)$ at $a = 0$ to be

$$T_2(x) = \frac{f''(0)}{2}(x-0)^2 + f'(0)(x-0) + f(0) = \frac{-1}{8}x^2 + \frac{1}{2}x + 1\,.$$

(The capital T stands for "Taylor", with the "2" representing the second derivative.) Figure 1 contains the graphs for the functions $f(x) = \sqrt{1+x}$, $L(x)$, and $T_2(x)$ — you see that $T_2(x)$ is a parabola that is tangent to $f(x)$ at the point $(0, 1)$. When we substitute $x = 1$ into $T_2(x)$, we obtain $T_2(1) = 1.375$, which is a better approximation.

Step 3: Third-order approximation. Going one step further, we can define a third-order approximation to be

$$T_3(x) = \frac{f^{(3)}(0)}{3\cdot 2}(x-0)^3 + \frac{f''(0)}{2}(x-0)^2 + f'(0)(x-0) + f(0) = \frac{1}{16}x^3 + \frac{-1}{8}x^2 + \frac{1}{2}x + 1\,.$$

If we evaluate this at $x = 1$, we get $T_3(1) = 1.4375$, which is an even better estimate of $\sqrt{2}$. Wow! Why is this working?

2. Approximation by Taylor Polynomials

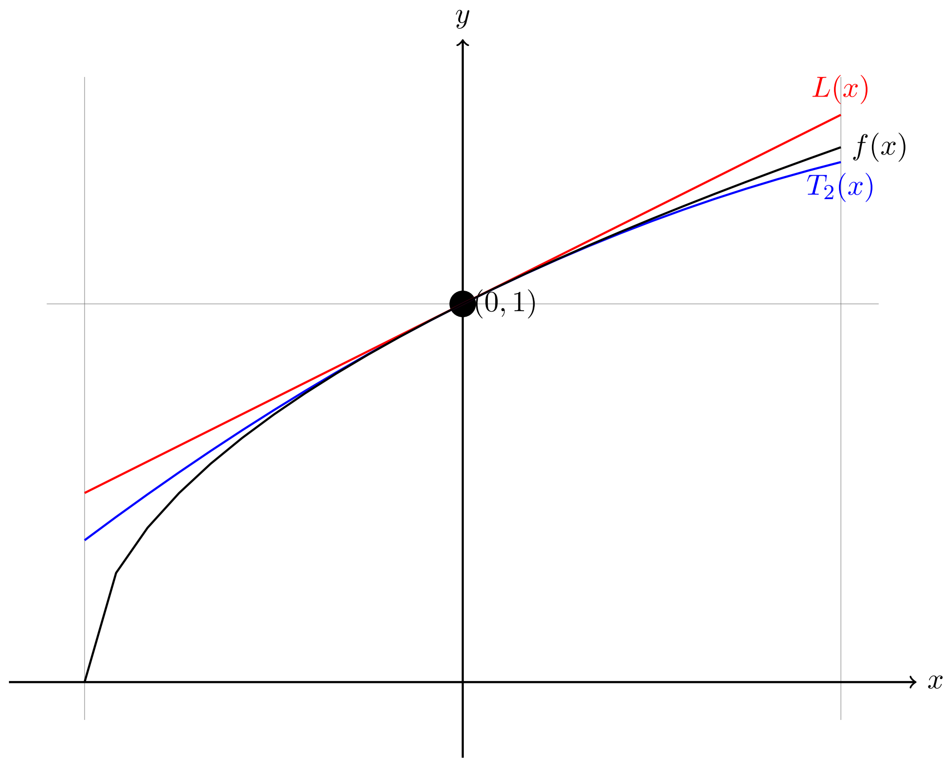
How did we come up with the formulas for $T_2(x)$ and $T_3(x)$ in our example above? To motivate our definitions, we use derivatives. To illustrate how this works, suppose that we have a function $f(x)$ and a degree four polynomial that approximates it near the point a, i.e.

$$f(x) \approx c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4\,.$$

Note that setting $x = a$, we obtain

$$f(a) \approx c_0,$$

since all of the terms involving $(x - a)$ will become zero. So, we know the constant term should be roughly $f(a)$.



Further, we can imagine that since the function and the polynomial are approximately the same, their derivatives are close as well. Suspending disbelief and hoping that this is true, we have

$$f'(x) \approx c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3\,.$$

Setting $x = a$ again, we have

$$f'(a) \approx c_1\,.$$

Continuing to take derivatives and evaluate at $x = a$, we obtain

$$\begin{aligned} f''(a) &\approx 2c_2 \\ f^{(3)}(a) &\approx 3 \cdot 2c_3 \\ f^{(4)}(a) &\approx 4 \cdot 3 \cdot 2c_4 \end{aligned}$$

A pattern is emerging! Solving for c_2, c_3, and c_4, it appears that we should expect

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3 \cdot 2}(x-a)^3 + \frac{f^{(4)}(a)}{4 \cdot 3 \cdot 2}(x-a)^4\,.$$

It is an amazing fact that for many functions, this type of approximation holds.

Definition 2.1. Let $f(x)$ be infinitely differentiable on an open interval I. Let a be in I. Define the *n-th Taylor polynomial for $f(x)$* to be

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3 \cdot 2}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n \cdot (n-1) \cdots 4 \cdot 3 \cdot 2}(x-a)^n\,.$$

IMPORTANT POINT: For many functions that we use in math and science, Taylor polynomials serve as good approximations for the functions. However, there are functions which are not approximated well by their Taylor polynomials. It took mathematicians over 100 years to discover this fact, so it is a subtle point. The following theorem provides a few functions with which we are free to use Taylor polynomials without concern for this subtlety.

Theorem 2.2. Suppose that $f(x)$ is one of the following functions:

- any polynomial
- e^x
- $\sin(x)$ or $\cos(x)$
- $(1+x)^k$ for some real number k

For the first three functions, for any choice of a defining $T_n(x)$ and for any value of x, we have

$$\lim_{n\to\infty} T_n(x) = f(x)\,.$$

For the fourth function, if we choose $a = 0$ and any x in $(-1, 1]$, the limit will hold. Thus, we can use $T_n(x)$ to approximate[1] $f(x)$ for these functions.

3. Examples

We now use this theorem to find some approximations for values that are both familiar and unfamiliar.

Example 3.1. Returning to our original example $f(x) = \sqrt{1+x}$, setting $a = 0$ and computing the first six derivatives gives us

$$\sqrt{1+x} \approx T_6(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6\,.$$

Doing so yields

$$\sqrt{2} \approx \frac{1439}{1024} = 1.4052734375\,.$$

Example 3.2. Since $\frac{d^n}{dx^n}(e^x) = e^x$ and $e^0 = 1$, we can set $a = 0$ and obtain

$$e^x \approx T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots + \frac{1}{n\cdot(n-1)\cdots 4\cdot 3\cdot 2}x^n\,.$$

From this it follows that

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{n\cdot(n-1)\cdots 4\cdot 3\cdot 2}\,.$$

Setting $n = 5$ we obtain $e \approx \frac{163}{60} = 2.716666\ldots$ which is correct to two digits.

[1]The limit condition we have given here is known as *pointwise convergence*. There is a stronger notion of approximation of functions called *uniform convergence* which is both incredibly important and more complicated.

Example 3.3. The functions $\sin(x)$ and $\cos(x)$ are particularly nice, because their derivatives have a repeating pattern. Thus, we have that

$$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)\cdot(2n)\cdots 3\cdot 2},$$

and $\cos(x)$ has a similar approximation. From this, we can estimate

$$\sin(1) \approx T_9(1) = 305353/362880 = 0.8414\ldots.$$

Example 3.4. Suppose you want to estimate $\int_0^1 \sin(x^2)\,dx$. We don't know how to find a nice anti-derivative for $\sin(x^2)$, so instead we use a Taylor polynomial with $a = 0$. We can obtain a Taylor polynomial for this function by substituting x^2 for x in a Taylor polynomial for $\sin(x)$; here we've used a seventh-order Taylor polynomial for $\sin(x)$:

$$\sin(x^2) \approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}$$

Thus, we have

$$\int_0^1 \sin(x^2)\,dx \approx \int_0^1 \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}\right) dx = 258019/831600 \approx 0.31027\,.$$

WolframAlpha computes that

$$\int_0^1 \sin(x^2)\,dx \approx 0.310268\ldots,$$

thus our approximation using a Taylor polynomial is quite accurate!