MA 114 — Calculus II   Fall 2014
Sections 1 – 8 and 401, 402

Exam 1   Sep. 23, 2014

Name: ______________________

Section: ______________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:** Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E

Exam Scores

Question	Score	Total
MC		20
5		15
6		16
7		16
8		15
9		18
Total		100

Unsupported answers for the free response questions may not receive credit!

Record the correct answer to the following problems on the front page of this exam.

1. Let $a > 0$ be a fixed number. Evaluate the improper integral $\int_a^{\infty} x^2 e^{-x^3}\,dx$.

A. ∞.

B. 0.

C. $\frac{1}{3e^{a^3}}$.

D. e^{a^3}.

E. $-\frac{1}{e^{a^3}}$.

$$\lim_{R\to\infty} \int_a^R x^2 e^{-x^3}\,dx = \lim_{R\to\infty} \frac{1}{3}\int_{a^3}^{R^3} e^{-u}\,du$$

$u = x^3$
$du = 3x^2\,dx$

$$= \lim_{R\to\infty} \left(-\frac{1}{3}\left(\underbrace{e^{-R^3}}_{\to 0} - e^{-a^3}\right)\right) = \frac{1}{3e^{a^3}}$$

2. Let $C > 1$ be a fixed number. Which of the following answers is true for the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{Cn+17}$?

$\frac{(-1)^n n}{Cn+17}$ does not go to zero.

A. The series is divergent.

B. The series is absolutely convergent.

C. The series is convergent, but not absolutely convergent.

D. The series is absolutely convergent, but not convergent.

E. None of the above.

Record the correct answer to the following problems on the front page of this exam.

3. Which of the following are true for a series $\sum_{n=1}^{\infty} a_n$? Check all that apply.

A. If the series is convergent, then it is also absolutely convergent.

B. If $\lim_{n\to\infty} a_n = 0$, then the series converges.

C. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

D. If the series is alternating, then it is convergent.

E. None of the above.

4. Evaluate the series $\sum_{n=0}^{\infty} 2^{3-2n}$.

$$= \sum 2^3 \cdot (2^{-2})^n = 8 \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= 8 \cdot \frac{1}{1-\frac{1}{4}} = \frac{32}{3}$$

A. The series is divergent.

B. $\sum_{n=0}^{\infty} 2^{3-2n} = 6.$

C. $\sum_{n=0}^{\infty} 2^{3-2n} = 11.$

D. $\sum_{n=0}^{\infty} 2^{3-2n} = \frac{32}{3}.$

E. $\sum_{n=0}^{\infty} 2^{3-2n} = \frac{21}{2}.$

Free Response Questions: Show your work!

5. Evaluate the integral
$$\int_2^{10} \frac{x}{\sqrt{x^2-4}}\,dx.$$

(2) The integral is improper b/c the function has an infinite discontinuity at 2. This is the only discontinuity on the interval $[2, 10]$.

(5) We first compute

$$\int_R^{10} \frac{x}{\sqrt{x^2-4}}\,dx \;\xrightarrow[du=2x\,dx]{u=x^2-4}\; \frac{1}{2}\int_{R^2-4}^{96} \frac{1}{\sqrt{u}}\,du$$

$$= \left(u^{\frac{1}{2}}\right)\Big|_{R^2-4}^{96} = \sqrt{96}-\sqrt{R^2-4}$$

(8) Now

$$\int_2^{10} \frac{x}{\sqrt{x^2-4}}\,dx = \lim_{R\to 2^+}\int_R^{10} \frac{x}{\sqrt{x^2-4}}\,dx$$

$$= \lim_{R\to 2^+}\Big(\sqrt{96}-\underbrace{\sqrt{R^2-4}}_{\to 0}\Big) = \underline{\underline{\sqrt{96}}}$$

Free Response Questions: Show your work!

6. Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{3}{\ln(n+1)}$ converges.

Let $a_n = \frac{3}{\ln(n+1)}$. Then $a_n > 0$.

(1) Put $b_n = \frac{1}{n}$. Then

(2) $$\frac{a_n}{b_n} = \frac{3n}{\ln(n+1)}.$$

(4) Note that

$$\lim_{x\to\infty} \frac{3x}{\ln(x+1)} \overset{\boxed{\frac{\infty}{\infty}}}{=} \lim_{x\to\infty} \frac{3}{\frac{1}{x+1}}$$

$$= \lim_{x\to\infty} 3(x+1) = \infty.$$

(1) Hence $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

(3) Moreover, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series).

(5) Thus by <u>limit comparison test</u> also

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3}{\ln(n+1)} \text{ diverges.}$$

7. Determine whether the following series converges or diverges. Make sure to state all tests that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

1 [Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{3^{n+1}}(n+1)^{\cancel{2}} \cdot \cancel{n!}}{\cancel{(n+1)!}\,\cancel{3^n}\, n^2}\right|$ $\cancel{(n+1)}$

4 [$= \left|\frac{3(n+1)}{n^2}\right| = \frac{3(n+1)}{n^2} \xrightarrow[n\to\infty]{} 0$

3 [Thus the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5+3^n}{100+4^n}$

2 [We have $\frac{5+3^n}{100+4^n} \leq \frac{5+3^n}{4^n} = \frac{5}{4^n} + \left(\frac{3}{4}\right)^n$

2 [Next, $\sum_{n=1}^{\infty} \frac{5}{4^n} = \sum 5\left(\frac{1}{4}\right)^n$ converges b/c it is a geometric series with $r = \frac{1}{4}$.

1 [Also $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ converges b/c we have again a geometric series $(r = \frac{3}{4})$.

1 [Hence the series $\sum \frac{5+3^n}{4^n} = \sum \frac{5}{4^n} + \sum\left(\frac{3}{4}\right)^n$ converges.

2 [By the comparison test also $\sum \frac{5+3^n}{100+4^n}$ converges.

Free Response Questions: Show your work!

8. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is absolutely convergent, conditionally convergent or divergent. Make sure to state all tests that you use.

(1) [Convergence by Leibniz:

(4) [$a_n = \frac{1}{\sqrt[3]{n}}$. Then $a_n > 0$ and $\{a_n\}$ is decreasing b/c $f(x) = \frac{1}{x^{1/3}}$ is decreasing.

Moreover, $\lim_{n \to \infty} a_n = 0$.

(3) [By Leibniz the series $\sum \frac{(-1)^n}{\sqrt[3]{n}}$ converges.

[Absolute convergence?

(4) [$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

This is a p-series with $p < 1$ and therefore the series diverges.

(3) [Hence the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges conditionally.

Free Response Questions: Show your work!

9. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 4^n}$.

(a) Find the radius of convergence.

(1) Ratio Test: Let $a_n = \frac{x^n}{n \cdot 4^n}$. Then

(4) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1} \cdot n \cdot 4^n}{(n+1)4^{n+1} \cdot x^n}\right| = \left|\frac{1}{4}\underbrace{\frac{n}{n+1}}_{\to 1} \cdot x\right|$

$\xrightarrow[n \to \infty]{} \left|\frac{1}{4}x\right|$. Now

(4) $\left|\frac{1}{4}x\right| < 1 \iff |x| < 4$, so $\boxed{\text{RoC} = 4}$.

(b) Find the interval of convergence.

We know from (a) that the series converges on $(-4, 4)$. It remains to test the endpoints.

(3) $\underline{x = -4}$: $\sum_{n=1}^{\infty} \frac{(-4)^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges b/c it is the alternating harmonic series.

(3) $\underline{x = 4}$: $\sum_{n=1}^{\infty} \frac{4^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges b/c it is the harmonic series.

(3) Hence the interval of convergence is $[-4, 4)$.