MA 114 — Calculus II Fall 2015
Exam 1 Sep. 22, 2015

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Signature: ______________________

Section: ______________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:**
  Record your answers on the upper right of this cover page by marking an X in the box corresponding to the correct answer. Your work will not normally be examined.
- **Detailed Solution Problems:**
  Show the reasoning you used to arrive at your answer in a clear and organized way. If you use a theorem or a test, state its name and show all the work required to apply it.

Multiple Choice Answers

Question					
1	A	~~B~~	C	D	E
2	A	B	C	D	~~E~~
3	A	B	~~C~~	D	E
4	A	~~B~~	C	D	E
5	A	~~B~~	C	D	E
6	~~A~~	B	C	D	E
7	A	B	C	~~D~~	E

Exam Scores

Question	Score	Total
MC		28
8		15
9		14
10		14
11		15
12		14
Total		100

Record the correct answer to the following problems on the front page of this exam.

1. If S_n is the nth partial sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the value of $S_{100} - S_{98}$ is

A. $\frac{1}{100}$;

B. $\frac{-1}{9900}$

C. $\frac{198}{9800}$

D. $\frac{199}{9900}$

E. $\frac{-2}{9800}$

$$a_n = \frac{(-1)^n}{n}$$

$$S_{100} - S_{98} = a_{99} + a_{100}$$

$$= -\frac{1}{99} + \frac{1}{100}$$

$$= \frac{-100}{9900} + \frac{99}{9900} = -\frac{1}{9900}$$

2. Let $L = \lim_{n\to\infty} n \sin\left(\frac{2}{n}\right)$. Which one of the following is true?

A. $L = \infty$

B. $L = \frac{\pi}{90}$

C. $L = \frac{1}{2}$

D. $L = 0$

E. $L = 2$

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ Let $x = \frac{2}{n}$

As $n \to \infty$, $x \to 0$ so

$$\lim_{n\to\infty} n \sin\left(\frac{2}{n}\right) = \lim_{x\to 0} \frac{2}{x}\sin(x)$$

$$= 2$$

3. Which one of the following is **not** true for every bounded sequence $\{a_n\}$? There is only one correct answer.

A. There exist numbers A and B satisfying $A \le a_n \le B$ for all n.

B. The sequence $\{\frac{a_n}{n}\}$ converges.

C. The sequence is monotonic.

D. The sequence $\{a_n^2\}$ is bounded.

E. If $\{a_n\}$ is decreasing then it is convergent.

Not all bounded sequences are monotonic, eg., $\{(-1)^n\}$

Record the correct answer to the following problems on the front page of this exam.

4. It is known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Thus the value of $\sum_{n=3}^{\infty} \frac{12}{n^2}$ is

A. $2\pi^2$

B. $2\pi^2 - 15$

C. $2\pi^2 + 15$

D. $2\pi^2 - 5/4$

E. none of the above

$$\sum_{n=3}^{\infty} \frac{12}{n^2} = 12\left(\sum_{n=3}^{\infty} \frac{1}{n^2}\right)$$

$$= 12\left(\sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - \frac{1}{4}\right) = 12\left(\frac{\pi^2}{6} - \frac{5}{4}\right)$$

$$= 2\pi^2 - 15$$

5. Which of the following is true for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^2+1}}$? There is only one correct answer.

A. The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

D. The sequence of partial sums is increasing.

E. The sequence of partial sums is decreasing.

$$\left|\frac{(-1)^{n-1}}{\sqrt{n^2+1}}\right| \geq \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{\sqrt{4n^2}} = \frac{1}{2n}$$

So (A) is false by the comparison test. Also, if $a_n = \frac{1}{\sqrt{n^2+1}}$, $\{a_n\}$ is a decreasing sequence with $\lim_{n\to\infty} a_n = 0$ and $a_n > 0$, so $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges by Leibniz test.

6. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^n}{n+1}(x-3)^n$.

A. $\frac{1}{2}$.

B. 2.

C. $\frac{5}{2}$.

D. 1.

E. ∞.

Let $a_n = \frac{2^n (x-3)^n}{n+1}$.

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(x-3)^{n+1}}{n+2} \frac{n+1}{2^n(x-3)^n}$$

$$= \frac{2(x-3)(n+1)}{n+2} = 2(x-3)\frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2|x-3| < 1$ for convergence, $L > 1$ for divergence. Hence series converges when $|x-3| < \frac{1}{2}$ and diverges when $|x-3| > \frac{1}{2}$. i.e. $R = \frac{1}{2}$

Record the correct answer to the following problems on the front page of this exam.

7. Apply the ratio test to determine convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3n}{n^2+4}$.

A. The series converges.

B. The series diverges.

C. The series is alternating.

D. The ratio test is inconclusive.

E. None of the above.

$$a_n = \frac{3n}{n^2+4}$$

$$\frac{a_{n+1}}{a_n} = \frac{3(n+1)}{[(n+1)^2+4]} \quad \frac{n^2+4}{3n}$$

$$= \frac{n+1}{n} \quad \frac{n^2+4}{[(n+1)^2+4]}$$

$$= \left(1+\frac{1}{n}\right) \quad \frac{1+\frac{4}{n^2}}{\left[\left(1+\frac{1}{n}\right)^2+\frac{4}{n^2}\right]}$$

Thus $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= (1+0)\left[\frac{1+0}{(1+0)^2+0}\right] = 1 \quad \text{inconclusive}$$

8. This problem considers the geometric series $\sum_{n=1}^{\infty} \frac{3x^{2n-1}}{4^n}$.

(a) Let S_n be the nth partial sum of the series. Give S_4.

$$S_4 = \frac{3x}{4} + \frac{3x^3}{4^2} + \frac{3x^5}{4^3} + \frac{3x^7}{4^4}$$

(b) Give the first term a of the series and find the common ratio r of the terms.

$a = \underline{\frac{3x}{4}}$ $\quad r = \underline{\frac{x^2}{4}}$

By inspection or $a_n = \frac{3x^{2n-1}}{4^n}$ so

$$r = \frac{a_{n+1}}{a_n} = \frac{3x^{2n+1}}{4^{n+1}} \cdot \frac{4^n}{3x^{2n-1}} = \frac{x^2}{4}.$$

(c) Find the sum of the geometric series.

$$S = \frac{a}{1-r} = \frac{\frac{3x}{4}}{1-\frac{x^2}{4}} = \frac{3x}{4-x^2}$$

(d) Specify all x for which the series converges $\underline{|x| < 2}$.

$\left|\frac{x^2}{4}\right| < 1$ i.e. $|x|^2 < 4$ i.e. $|x| < 2$

9. This problem considers the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n+n^3}}$.

(a) Establish a suitable inequality and apply the comparison test to determine whether the series **converges absolutely**. Show all work required to apply this test.

Let $a_n = \dfrac{\sin(n)}{\sqrt{n+n^3}}$. Then since $|\sin(n)| \le 1$,

$|a_n| \le \dfrac{1}{\sqrt{n+n^3}} \le \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges since it is a p-series with $p = \frac{3}{2} > 1$. Hence $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges by the comparison test. Thus the series converges absolutely.

(b) Apply a test or theorem to determine whether the series converges or diverges. State the test or theorem and show all work required to apply it.

Every series that converges absolutely, converges. Thus the given series converges by part (a).

10. Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{(n+1)^2}$ converges or diverges. Make sure to state all tests that you use and to show all work required to apply the test.

Let $a_n = \frac{n}{(n+1)^2}$. Then $a_n \geq 0$ for all n

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{n}}{(1+\frac{1}{n})^2} = \frac{0}{1^2} = 0.$

Also $a_n = f(n)$, where $f(x) = \frac{x}{(x+1)^2}$,

and

$$f'(x) = \frac{(x+1)^2 \cdot 1 - x[2(x+1)]}{(x+1)^4} = \frac{(x+1)-2x}{(x+1)^3}$$

$$= \frac{1-x}{(x+1)^3} < 0 \text{ for } x > 1.$$

Hence f is decreasing on $[1,\infty)$ so $\{a_n\}$ is a decreasing sequence.

Thus $\sum_{n=1}^{\infty}(-1)^n a_n$ converges by the Leibnitz test.

11. Consider the power series $\sum_{n=0}^{\infty} \frac{(x-7)^n}{5^n(n+3)}$.

(a) Find the radius of convergence 5.

Let $a_n = \frac{(x-7)^n}{5^n(n+3)}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(x-7)^{n+1}}{5^{n+1}(n+4)} \frac{5^n(n+3)}{(x-7)^n} = \frac{1}{5}(x-7)\frac{n+3}{n+4}$$

$$= \frac{x-7}{5} \frac{1+\frac{3}{n}}{1+\frac{4}{n}} \text{ so } L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-7|}{5}$$

By the ratio test, the series converges if $L<1$, i.e., $|x-7|<5$ and diverges if $L>1$, i.e., $|x-7|>5$. Hence $R=5$

(b) Find the interval of convergence $2 \le x < 12$. (You will need to check the endpoints.)

If $x-7=5$, then the series is $\sum_{n=0}^{\infty} \frac{1}{n+3}$ $= \sum_{n=3}^{\infty} \frac{1}{n}$, which diverges since the harmonic series diverges.

If $x-7=-5$, then the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$.

Let $a_n = \frac{1}{n+3}$. Then $a_n \ge 0$ and $a_{n+1} = \frac{1}{n+4} < \frac{1}{n+3} = a_n$ for all n and $\lim_{n\to\infty} a_n = 0$. Hence $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ converges.

Thus the given series converges exactly when $-5 \le x-7 < 5$, i.e., $2 \le x \le 12$.

12. Represent each of the functions below as a power series and find the radius of convergence.

(a) $f(x) = \dfrac{x^2}{1+3x}$. Radius of convergence $\frac{1}{3}$.

$$f(x) = \frac{x^2}{1-(-3x)} = x^2 \sum_{n=0}^{\infty} (-3x)^n = x^2 \sum_{n=0}^{\infty} (-1)^n 3^n x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n 3^n x^{n+2} = \sum_{n=2}^{\infty} (-1)^n 3^{n-2} x^n$$

This converges only for $|-3x| < 1$, i.e., $|x| < \frac{1}{3}$.

(b) $f(x) = \dfrac{2x}{(1-x^2)^2}$. Radius of convergence 1.

Hint: $f(x) = \dfrac{d}{dx} \dfrac{1}{1-x^2}$.

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$$ and this converges for $|x^2| < 1$, i.e., $|x| < 1$ so $R = 1$. Hence $f(x) =$

$$\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \sum_{n=1}^{\infty} 2n x^{2n-1}$$ and

$$\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = -\frac{1}{(1-x^2)^2}(-2x) = \frac{2x}{(1-x^2)^2}.$$

Thus $$\frac{2x}{(1-x^2)^2} = \sum_{n=1}^{\infty} 2n x^{2n-1}$$ and the radius of convergence R is the same.