Exam 2

Name: ______________________ Section and/or TA: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 12 multiple choice questions and 6 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. The wise student will show work for the multiple choice problems as well.

Multiple Choice Questions

1	(A)	(B)	(C)	(D)	(E)	**7**	(A)	(B)	(C)	(D)	(E)
2	(A)	(B)	(C)	(D)	(E)	**8**	(A)	(B)	(C)	(D)	(E)
3	(A)	(B)	(C)	(D)	(E)	**9**	(A)	(B)	(C)	(D)	(E)
4	(A)	(B)	(C)	(D)	(E)	**10**	(A)	(B)	(C)	(D)	(E)
5	(A)	(B)	(C)	(D)	(E)	**11**	(A)	(B)	(C)	(D)	(E)
6	(A)	(B)	(C)	(D)	(E)	**12**	(A)	(B)	(C)	(D)	(E)

SCORE

Multiple Choice	13	14	15	16	17	18	Total Score
36	12	12	15	5	4	16	100

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Multiple Choice Questions

1. Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$. If the ratio test is applied to the series, which of the following inequalities results, implying that the series converges?

   A. $\lim_{n\to\infty} \frac{e}{n!} < 1$

   B. $\lim_{n\to\infty} \frac{n!}{e} < 1$

   C. $\lim_{n\to\infty} \frac{n+1}{e} < 1$

   D. $\lim_{n\to\infty} \frac{e}{n+1} < 1$

   E. $\lim_{n\to\infty} \frac{e}{(n+1)!} < 1$

2. The interval of convergence of the power series $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$ is

   A. $[0]$

   B. $\left(-\frac{1}{3}, \frac{1}{3}\right)$

   C. $(-3, 3]$

   D. $(-3, 3)$

   E. $(-\infty, +\infty)$

3. The sum of the infinite geometric series $1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \cdots$ is

   A. $\frac{3}{5}$

   B. $\frac{2}{3}$

   C. $\frac{5}{3}$

   D. $\frac{3}{2}$

   E. $\frac{5}{2}$

4. Which of the following sequences converge?

   I. $\left\{ \frac{5n}{2n-1} \right\}$

   II. $\left\{ \frac{e^n}{n} \right\}$

   III. $\left\{ \frac{e^n}{1+e^n} \right\}$

   A. I only

   B. II only

   C. I and II only

   D. I and III only

   E. I, II, and III

5. If $\lim_{M\to\infty} \int_1^M \frac{dx}{x^p}$ converges, then which of the following must be true?

A. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

B. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

C. $\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges.

D. $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges.

E. $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ diverges.

6. A series $\sum a_n$ is convergent if and only if

A. the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is greater than 1.

B. its sequence of terms $\{a_n\}$ converges to 0.

C. its sequence of partial sums $\{S_n\}$ converges to some real number.

D. its sequence of terms $\{a_n\}$ is alternating.

E. its sequence of partial sums $\{S_n\}$ is bounded.

7. Which of the following statements is true? (There is only one.)

   A. If $0 \le b_n \le a_n$ and $\sum b_n$ converges then $\sum a_n$ converges.

   B. If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ is convergent.

   C. The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.

   D. If $\sum a_n$ is convergent for $a_n > 0$ then $\sum(-1)^n a_n$ is also convergent.

   E. The ratio test can be used to show that $\sum \frac{1}{n^{10}}$ converges.

8. Let S_N be the N-th partial sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1}.$$

Thus, $S_1 = 1$, $S_2 = \frac{2}{3}$. Compute $S_{50} - S_{49}$.

   A. $-\frac{1}{99}$.

   B. $-\frac{1}{50}$

   C. 1

   D. $\frac{2}{9603}$

   E. 0

9. Consider the series $\sum_{n=1}^{\infty} \frac{3}{4^n + 6n - 4}$. Applying the comparison test with the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$ leads to the following conclusion.

   A. The test is inconclusive.

   B. The series converges absolutely.

   C. The series converges conditionally.

   D. The series diverges.

   E. The test cannot be applied to $a_n = \frac{3}{4^n + 6n - 4}$ and $b_n = \frac{3}{4^n}$.

10. The radius of convergence for the series $\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n}$ is

   A. 1

   B. 1/10

   C. 10

   D. $n/10$

   E. ∞

11. The series $\sum_{n=0}^{\infty} \frac{n^2+1}{n^4+1}$

A. converges by the Ratio Test.

B. diverges by the Integral Test.

C. converges by the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

D. diverges by the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

E. diverges because it does not alternate in sign.

12. The series $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2}$ is

A. converges absolutely.

B. converges conditionally.

C. diverges.

D. eventually oscillates between -1 and 1, but never converges.

E. none of the above.

Free Response Questions

13. Find the first four (4) terms of each of the following sequences.

(a) (6 points) $a_n = \dfrac{1}{(n+1)!}$

Solution:

$$1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}$$

(b) (6 points) $a_1 = 2$ and $a_{n+1} = \dfrac{1}{3 - a_n}$

Solution:

$$2, 1, \frac{1}{2}, \frac{2}{5}, \frac{5}{13}$$

14. Determine if the sequence is convergent or divergent. If convergent give its limit.

(a) (4 points) $a_n = \dfrac{n+1}{3n-1}$

Solution: The sequence converges.

$$\lim_{n\to\infty} \frac{n+1}{3n-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3-\frac{1}{n}} = \frac{1}{3}.$$

(b) (4 points) $a_n = n^2 e^{-n}$

Solution: The sequence converges.

$$\lim_{n\to\infty} \frac{n^2}{e^n} = 0.$$

(c) (4 points) $a_n = \dfrac{3^n}{2^n}$

Solution: The sequence diverges.

$$\lim_{n\to\infty} \frac{3^n}{2^n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = +\infty.$$

15. Determine the convergence or divergence of each of the following series. State clearly what test you used and show your work.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution: This series diverges by the p-series test with $p = 1/2$.

(b) (5 points) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3}$

Solution:

$$\sin^2(n) \leq 1$$
$$\frac{\sin^2(n)}{n^3} \leq \frac{1}{n^3}$$

The latter series converges by the p-series test with $p = 3$, so the given series converges by the Comparison Test with the series $\sum \frac{1}{n^3}$.

(c) (5 points) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Solution: Use the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}$$
$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \frac{1}{2} = \frac{1}{2} < 1$$

Since the limit is less than 1, the series converges by the Ratio Test.

16. (5 points) Use the integral test to determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$$

converges or diverges. Show your work and clearly state your answer.

Solution: Let $f(x) = \dfrac{1}{x\ln x}$. $f'(x) = -\dfrac{\ln x + 1}{x^2(\ln x)^2)} < 0$ for $x > 2$ so the function is decreasing. Let $u = \ln x$ then $du = \dfrac{dx}{x}$ and

$$\begin{aligned} int_2^{\infty}\frac{1}{x\ln x}dx &= \lim_{M\to\infty}\int_2^M \frac{1}{x\ln x}\,dx \\ &= \lim_{M\to\infty}\int_{\ln 2}^M \frac{1}{u}\,du \\ &= \lim_{M\to\infty} \ln u|_{\ln 2}^M \\ &= \text{diverges} \end{aligned}$$

Since the integral diverges, then the series also diverges.

17. (4 points) Use the comparison test to determine whether the series

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

converges or diverges.

Solution: Let $b_k = 1/k$ and $a_k = (\ln k)/k$. For $k \geq 3$, $\ln k \geq 1$ so $a_k \geq b_k$. Since $\sum_{k=1}^{\infty} b_k$ is the (divergent) harnonic series (that is, the p-series with $p = 1$), this series diverges.

18. A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \frac{4}{3^4}x^3 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots = \sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}}x^n.$$

for all x in the interval of convergence for the power series.

(a) (4 points) Find the radius of convergence for the power series. ***Show your work.***

Solution: Apply the ratio test with

$$a_n = \frac{n+1}{3^{n+1}}x^n.$$

We get

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+2}{3^{n+2}} \cdot \frac{3^{n+1}}{n+1}\right| |x| = \frac{1}{3}|x|$$

so the radius of convergence is $R = 3$.

(b) (4 points) Find the interval of convergence for the power series. ***Show your work.***

Solution: We check $x = \pm 3$. If $x = -3$ we get the numerical series

$$\sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}}(-3)^n = \sum_{n=0}^{\infty} (-1)^n (n+1)\frac{1}{3}.$$

This series diverges by the divergent series test since $\lim_{n\to\infty}(-1)^n(n+1)\frac{1}{3}$ does not exist.

For $x = +3$ a similar analysis shows that the series diverges. Hence, the interval of convergence is $(-3, 3)$.

(c) (4 points) Find the power series representation for $f'(x)$ and state its radius of convergence.

Solution: We can differentiate term-by-term to get

$$\sum_{n=1}^{\infty} \frac{n}{n+1} 3^{n+1} x^{n-1} = \sum_{n=0}^{\infty} \frac{(n+1)}{n+2} 3^{n+2} x^{n}$$

which converges absolutely for $|x| < 3$.

(d) (4 points) Find the power series representation for $\int f(x)\,dx$.

Solution: We can integrate term by term and get

$$\sum_{n=0}^{\infty} \frac{n+1}{3^{n+1}} \frac{x^{n+1}}{n+1} + C = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^{n+1} + C$$

which converges for $|x| < 3$.