Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Give the first three **nonzero** terms of the sequence $\{a_1, a_2, \ldots\}$ defined by

$$a_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}.$$

   A. $\{\frac{1}{1}, \frac{-1}{4}, \frac{-1}{16}\}$
   B. $\{\frac{1}{1}, \frac{1}{9}, \frac{1}{25}\}$
   C. $\{\frac{-1}{1}, \frac{-1}{9}, \frac{-1}{25}\}$
   D. $\{\frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}\}$
   E. $\{\frac{1}{1}, \frac{-1}{27}, \frac{1}{125}\}$

2. (5 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n + n}$ converge or diverge?

   A. Converges by the limit comparison test to $\sum_{n=1}^{\infty} \frac{1}{3^n}$.
   B. Converges because $\displaystyle\lim_{n\to\infty} \frac{1}{3^n + n} = 0$.
   C. Converges because it is a geometric series and $|r| < 1$.
   D. Diverges by a comparison test to $\sum_{n=1}^{\infty} \frac{1}{3^n}$.
   E. Diverges by a comparison test to $\sum_{n=1}^{\infty} \frac{1}{n}$.

3. (5 points) Which of the following series converge?

   A. $\displaystyle\sum_{n=10}^{\infty} \frac{n}{\sqrt{n^2+1}}$

   B. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{(n+2)^{\frac{3}{2}}}$

   C. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$

   D. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2}$

   E. None of the above series converge.

4. (5 points) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n$

   A. $\frac{1}{3}$

   B. $\frac{1}{2}$

   C. 5

   D. 2

   E. This series is divergent.

5. (5 points) What would you compare $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n^3+2n+1}}{n^3-1}$ to for a conclusive limit comparison test?

A. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$

B. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$

C. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

D. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3}$

E. The limit comparison test can't be used to understand convergence for this series.

6. (5 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n+1)!}$ converge or diverge?

A. Diverges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1$

B. Converges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{n+1}{(2n+3)} = \frac{1}{2}$

C. Diverges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{(2n+3)(2n+2)}{n} = \infty$

D. Converges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{n+1}{n(2n+3)(2n+2)} = 0$

E. Diverges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{n(2n+3)}{n+1} = \infty$

7. (5 points) Find the smallest value of N so that S_N approximates $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ to within an error of at most .01.

   A. $N = 2$

   B. $N = 6$

   C. $N = 10$

   D. $N = 4$

   E. $N = 3$

8. (5 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n(x-5)^n}{7^n}$?

   A. $(-7, 7)$

   B. $(-2, 12)$

   C. $(4, 6)$

   D. $(6, 8)$

   E. $(-\infty, \infty)$

9. (5 points) Which power series represents the function $\sin(5x^2)$ on the interval $(-\infty, \infty)$?

A. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5^{2n}(2n)!} x^{2n}$

B. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n+1)!} x^{4n+2}$

C. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n+1)} x^{4n+2}$

D. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^{2n}} x^{4n}$

E. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!} x^{4n}$

10. (5 points) Find the first 3 **nonzero** terms of the Taylor series for $f(x) = xe^{-x}$ centered at 0.

A. $x - x^2 + \frac{1}{2}x^3$

B. $1 - x^2 + \frac{1}{2}x^4$

C. $1 + x + \frac{1}{2}x^2$

D. $x - \frac{1}{2}x^2 + \frac{1}{6}x^3$

E. $1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$

Free Response Questions

11. (a) (5 points) Decide if the series converges or diverges (Clearly state which test(s) are used):

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Solution: Converges by ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{1}{2} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{2} < 1$$

(b) (5 points) Decide if the series converges or diverges (Clearly state which test(s) are used):

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n+3^n}$$

Solution: The absolute value series converges by comparison with a convergent geometric series:

$$\left| \frac{(-2)^n}{n+3^n} \right| = \frac{2^n}{n+3^n} < \frac{2^n}{3^n} = \left(\frac{2}{3} \right)^n, \qquad \frac{2}{3} < 1,$$

therefore the series converges.

12. Are the series below **absolutely convergent, conditionally convergent,** or **divergent**? Justify your answer.

(a) (6 points)

$$\sum_{n=1}^{\infty}(-1)^n\frac{n+2}{n^2+3}$$

Solution: The series converges conditionally by the alternating series test:

$$\lim_{n\to\infty}\frac{n+2}{n^2+3} = \quad \text{(L'Hopital's rule)} \quad \lim_{n\to\infty}\frac{1}{2n} = 0,$$

$\dfrac{n+2}{n^2+3} > \dfrac{(n+1)+2}{(n+1)^2+3}$ because $\left(\frac{x+2}{x^2+3}\right)' = \frac{-x^2-4x+3}{(x^2+3)^2}$ is negative for $x \geq 1$.

However, the absolute value series diverges by limit comparison with the p-series $\sum_{n=1}^{\infty}\frac{1}{n}$:

$$\lim_{n\to\infty}\frac{n}{1}\frac{n+2}{n^2+3} = \lim_{n\to\infty}\frac{n^2+2n}{n^2+3} = \quad \text{(L'Hopital's rule)} \lim_{n\to\infty}\frac{2n+2}{2n} = 1 > 0.$$

(b) (4 points)

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$$

Solution: The series converges absolutely by the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1.$$

13. (a) (5 points) What is the **radius** of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^3}{9^n}\right)x^n$?

Solution: By ratio test, this series converges if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{9^{n+1}}|x|^{n+1}\frac{9^n}{n^3}\frac{1}{|x|^n} = \frac{|x|}{9}\lim_{n\to\infty}\frac{(n+1)^3}{n^3} = \frac{|x|}{9} < 1.$$

and diverges if $\frac{|x|}{9} > 1$. The radius of convergence is 9.

(b) (5 points) For which x does $\displaystyle\sum_{n=1}^{\infty}(5x)^n$ converge? (i.e. find the interval of convergence.)

Solution: This is a geometric series, so it converges if and only if $|5x| < 1$. The interval of convergence is $(-\frac{1}{5}, \frac{1}{5})$.

14. (a) (4 points) Find the first six derivatives of $f(x) = \cos(x)$ and evaluate each at $a = \frac{\pi}{2}$.

Solution: $f^{(0)}(x) = \cos(x), f^{(1)}(x) = -\sin(x), f^{(2)}(x) = -\cos(x), f^{(3)}(x) = \sin(x), f^{(4)}(x) = \cos(x), f^{(5)}(x) = -\sin(x), f^{(6)}(x) = -\cos(x)$.

$\cos(\frac{\pi}{2}) = 0, \sin(\frac{\pi}{2}) = 1$

$f^{(0)}(\frac{\pi}{2}) = 0, f^{(1)}(\frac{\pi}{2}) = -1, f^{(2)}(\frac{\pi}{2}) = 0, f^{(3)}(\frac{\pi}{2}) = 1, f^{(4)}(\frac{\pi}{2}) = 0, f^{(5)}(\frac{\pi}{2}) = -1, f^{(6)}(\frac{\pi}{2}) = 0$.

(b) (6 points) Find the Taylor series expansion of $f(x) = \cos(x)$ about $a = \frac{\pi}{2}$. **(Note: this is not centered at $a = 0$.)**

Solution: $\cos(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x - \frac{\pi}{2})^n$

$f^{(2n)}(\dfrac{\pi}{2}) = 0$

$f^{(2n+1)}(\dfrac{\pi}{2}) = (-1)^{n+1}$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x - \frac{\pi}{2})^{2n+1}.$$

15. (a) (5 points) Write a series expansion for the function $f(x) = \dfrac{1}{1-x^2}$ centered at $x = 0$.

Solution: By the formula for geometric series:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$$

(b) (5 points) Use your answer in part (a) to find a series expansion for the function $g(x) = \dfrac{2x}{(1-x^2)^2}$ centered at 0. (**Hint:** It will help to find the derivative of $f(x)$ in part (a).)

Solution: $$\frac{2x}{(1-x^2)^2} = \left(\frac{1}{1-x^2}\right)' = \sum_{n=0}^{\infty}\left(x^{2n}\right)' = \sum_{n=0}^{\infty} 2nx^{2n-1}.$$