Record the correct answer to the following problem on the front page of this exam.

(1) The form of the partial fraction decomposition of the rational function

$$f(x) = \frac{3x+2}{(x+1)^2(x^2+3)},$$

with the parameters A, B, C, D, E being constants to be determined, is:

A) $\dfrac{A}{(x+1)^2} + \dfrac{Bx+C}{x^2+3}$

B) $\dfrac{A}{x+1} + \dfrac{Bx+C}{(x+1)^2} + \dfrac{D}{x^2+3}$

C) $\dfrac{A}{x+1} + \dfrac{Bx+C}{(x+1)^2} + \dfrac{Dx+E}{x^2+3}$

D) $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{x^2+3}$

E) none of the above

$$\frac{3x+2}{(x+1)^2(x^2+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+3)}$$

↑ repeated linear

↑ irreducible

(2) Which of the following statements is false?

A) $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges

B) $\displaystyle\int_0^1 \frac{dx}{x^{4/3}}$ diverges

C) $\displaystyle\int_1^2 \frac{dx}{(x-1)^2}$ diverges

D) $\displaystyle\int_2^4 \frac{dx}{x-2}$ converges

E) $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges

$$\int_2^4 \frac{dx}{x-2}$$

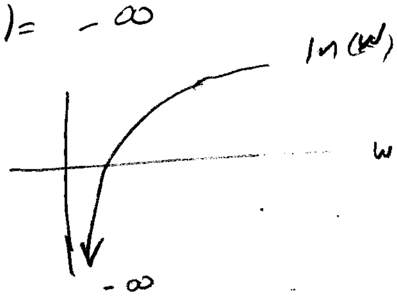
$$= \lim_{u \to 2^+} \int_u^4 \frac{dx}{x-2}$$

$$= \lim_{u \to 2^+} \ln(x-2) \Big|_u^4$$

$$= \ln(4-2) - \lim_{u \to 2^+} \ln(u-2)$$

$$= \ln(2) - (-\infty) \quad \text{diverges}$$

$$\lim_{w \to 0^+} \ln(w) = -\infty$$



Record the correct answer to the following problem on the front page of this exam.

(3) Which of the integrals below represents the length of the curve $y = \tan x$ from $x = 0$ to $x = \pi/4$?

A) $\int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx$

B) $\int_0^{\pi/4} \sqrt{1+\tan^2 x \sec^2 x}\, dx$

C) $\int_0^{\pi/4} \sqrt{1+\sec^4 x}\, dx$

D) $2\pi \int_0^{\pi/4} \tan x \sqrt{1+\tan^2 x}\, dx$

E) $\int_0^{\pi/4} x \tan x\, dx$

$y' = \sec^2(x)$

$(y')^2 = (\sec^2(x))^2$

$L = \int_a^b \sqrt{1+(y')^2}\, dx$

$= \int_0^{\pi/4} \sqrt{1+\sec^4 x}\, dx$

(4) A triangular laminar (thin plate) of uniform mass density has its vertices at the points $A = (0,3)$, $B = (0,0)$, and $C = (1,0)$ in the x-y plane. Where is the center of mass of the laminar located?

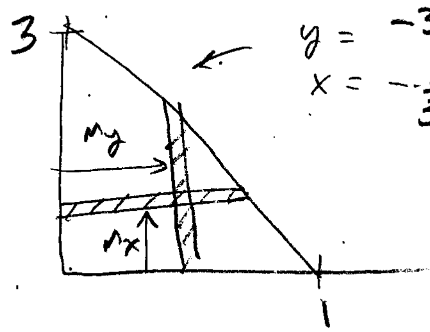
A) $(0,1)$

B) $(\frac{1}{3},1)$

C) $(\frac{1}{2},\frac{3}{2})$

D) $(\frac{2}{3},1)$

E) $(\frac{1}{3},\frac{3}{2})$



$y = -3x+3$

$x = -\frac{1}{3}y + 1$

$\rho(\frac{1}{2}bh)$

$\text{Mass} = \rho(\frac{1}{2})(3)(1)$

$= \frac{3\rho}{2}$

$M_x = \rho\int_0^3 y\left(-\frac{y}{3}+1\right) dy = \rho\int_0^3 \left(-\frac{y^2}{3}+y\right) dy = \rho\left(-\frac{y^3}{9}+\frac{y^2}{2}\right)\Big|_0^3 = \rho\left(-\frac{27}{9}+\frac{9}{2}\right) = \rho\left(\frac{3}{2}\right)$

$M_y = \rho\int_0^1 x(-3x+3)\, dx = \rho\int_0^1 (-3x^2+3x)\, dx = \rho\left(\frac{-3x^3}{3}+\frac{3x^2}{2}\right)\Big|_0^1 = \rho\left(\frac{1}{2}\right)$

$\text{COM}_y = \frac{3\rho}{2}\cdot\frac{2}{3\rho} = 1$

$\Rightarrow \text{COM} = (\frac{1}{3},1)$

$\text{COM}_x = \frac{\rho}{2}\cdot\frac{2}{3\rho} = \frac{1}{3}$

Record the correct answer to the following problem on the front page of this exam.

(5) Which of the following differential equations is NOT separable?

A) $xy' + y = y^2$
B) $(1+x^2)y' = x^3y$
C) $x(y^2-1) + y(x^2-1)y' = 0$
D) $y' = \sin y$
E) $y^2 + x^2y' = xyy'$

$x^2y' - xyy' = -y^2$

$(x^2 - xy)y' = -y^2$

$y' = \dfrac{-y^2}{x^2 - xy}$ ← can't simplify to $f(x)\,g(y)$

(6) Which of the following statements is false? In what follows, k and b are given constants, and C stands for an arbitrary constant.

A) The general solution of the differential equation $y' = k(y-b)$ is $y = b + Ce^{kt}$.
B) The general solution of the differential equation $y' = k(y-b)$ is $y = b - Ce^{kt}$.
C) The general solution of the differential equation $y' = -k(y-b)$ is $y = b + Ce^{-kt}$.
D) If $k > 0$, then all solutions of $y' = k(y-b)$ tend to ∞ as $t \to \infty$.
E) If $k > 0$, then all solutions of $y' = -k(y-b)$ approach the same limit as $t \to \infty$.

$k > 0$

$y = b + Ce^{kt}$

$C > 0$ $y \to \infty$ as $t \to \infty$

$\boxed{C < 0 \quad y \to -\infty \text{ as } t \to \infty}$

Free Response Questions: Show your work!

(7) Evaluate the integral
$$\int \frac{3x}{(x-1)(x^2+2)}dx.$$

$$\frac{3x}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

$$3x = A(x^2+2) + (Bx+C)(x-1)$$

$x=1$ $\quad 3 = A(3) \Rightarrow A = 1$

$x=0$ $\quad 0 = 1(2) + C(-1) \Rightarrow C = 2$

$x=-1$ $\quad -3 = 1(3) + (-B+2)(-2) \Rightarrow 0 = 2B+2 \Rightarrow B = -1$

$$= \int \left(\frac{1}{x-1} + \frac{-x+2}{x^2+2}\right)dx = \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+2}\,dx + 2\int \frac{dx}{x^2+2}$$

$u = x^2+2$

$\frac{du}{2} = x\,dx$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+2) + 2\int \frac{dx}{x^2+2}$$

$x = \sqrt{2}\tan\theta$

$x^2+2 = 2\sec^2\theta$ d

$dx = \sqrt{2}\sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+2) + \frac{2\sqrt{2}}{2}\int d\theta$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+2) + 2\sqrt{2}\,\theta + C$$

$$= \ln|x-1| + \frac{1}{2}\ln(x^2+2) + \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Free Response Questions: Show your work!

(8) Compute the surface area of the surface obtained by rotating the graph of $y = \sqrt{1+2x}$ about the x-axis over the interval $[0,1]$.

$$y' = \frac{1}{2}(1+2x)^{-1/2}(2) = (1+2x)^{-1/2}$$

$$(y')^2 = (1+2x)^{-1}$$

$$S = 2\pi \int_0^1 y\sqrt{1+(y')^2}\,dy = 2\pi \int_0^1 \sqrt{1+2x}\sqrt{1+\frac{1}{1+2x}}\,dx$$

note:
① $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$
② $\sqrt{a}\cdot\sqrt{b} \neq ab$
common error

$$= 2\pi \int_0^1 \sqrt{1+2x+1}\,dx$$

$$= 2\pi \int_0^1 \sqrt{2+2x}\,dx = 2\pi \int_0^1 (2+2x)^{1/2}$$

$u = 2+2x$
$du = 2\,dx$

$$= 2\pi\left(\frac{2}{3}\right)\frac{(2+2x)^{3/2}}{2}\Big|_0^1$$

$$= \frac{2\pi}{3}\left(4^{3/2} - 2^{3/2}\right) = \frac{2\pi}{3}(5.17157)$$

$$\approx 10.8313$$

Free Response Questions: Show your work!

(9) The following table gives the measured values of a force function $f(x)$, where x is in meters and $f(x)$ in newtons.

work = area under force curve

x	0	2	4	6	8
$f(x)$	10.0	9.5	9.3	9.1	9.2

$\Rightarrow \Delta x = 2\,m$

(a) Use Simpson's Rule to estimate the work done by the force f in moving an object from $x = 0$ to $x = 8$ meters.

$$A \approx \frac{\Delta x}{3}\left(f(x_0) + 4f(2) + 2f(4) + 4f(6) + f(8) \right)$$

$$= \frac{2}{3}\left(10.0 + 4(9.5) + 2(9.3) + 4(9.1) + 9.2 \right)$$

$$= \frac{2}{3}(112.2) = 74.8 \text{ Newtons}$$

(b) It is known that the force function $f(x)$ satisfies the inequality $|f^{(4)}(x)| \leq 2$ on the interval $[0, 8]$. Let S_N be the Nth approximation to $\int_0^8 f(x)dx$ by Simpson's rule. Use the given inequality on $|f^{(4)}(x)|$ to find the smallest N that guarantees $\text{Error}(S_N) \leq 10^{-1}$. (Hint: Use the error bound for S_N given on the last page of the exam.)

$$\text{Error}_N \leq \frac{K_4 (b-a)^5}{180 N^4} \leq \frac{|f^{(4)}(x)|(8-0)^5}{180 (N^4)}$$

$$\text{Error}_N \leq \frac{2(8)^5}{180 N^4} \leq 10^{-1}$$

$$\frac{2(8^5)}{180(10^{-1})} \leq N^4$$

$$N \geq \sqrt[4]{\frac{2(8^5)(10)}{180}} = 7.767$$

$\Rightarrow$ smallest $N = 8$

Free Response Questions: Show your work!

(10) Let $T_n(x)$ $(n = 0, 1, 2, \cdots)$ be the nth Taylor polynomial for $f(x) = e^x$ centered at $a = 0$.

(a) Find the Taylor polynomial $T_n(x)$.

$f(0) = e^0 = 1$

$f'(0) = e^0 = 1$

$f''(0) = e^0 = 1$

$\vdots$

$f^{(n)}(0) = e^0 = 1$

$f(x) = f(0) + f'(0)(x-0) + f''(0)\frac{(x-0)^2}{2} + \cdots$

$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

(b) Find a value of n for which

$$|e^x - T_n(x)| \le 10^{-2}$$

on the interval $[0, 1]$. (Hint: Use the error bound given on the last page of the exam.)

$f^{(k+1)}(x) = e^x$ on $[0,1]$ $\quad f^{(k+1)}(x)$ max when $x = 1$

$f^{(k+1)}(1) = e^1 = e$

$10^{-2} \le \frac{f^{(k+1)}(x)\,|1-0|^n}{(n+1)!}$

$10^{-2} \le \frac{e(1)}{(n+1)!}$

$(n+1)! \le e(100) = 271.83$

when $n = 6$ ← answer

$(n+1)! = 7! = 5040$

when $n = 5$

$(n+1)! = 6! = 720$ ← not big enough

(11) Solve the initial value problem

$$\frac{dx}{dt} = x^2(1-t^2), \qquad x(1) = 1.$$

$$\frac{dx}{x^2} = (1-t^2)\,dt$$

$$\int \frac{dx}{x^2} = \int (1-t^2)\,dt$$

$$\frac{-1}{x} = t - \frac{t^3}{3} + c$$

$x = 1$, $t = 1$

$$\frac{-1}{1} = 1 - \frac{1}{3} + c$$

$$-\frac{5}{3} = c$$

$$\frac{-1}{x} = t - \frac{t^3}{3} - 5/3$$

$$x = \frac{-1}{t - \frac{t^3}{3} - 5/3}$$