Exam 2

Name: ______________________ Section and/or TA: ______________

Last Four Digits of Student ID: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 A B C D E   **6** A B C D E

2 A B C D E   **7** A B C D E

3 A B C D E   **8** A B C D E

4 A B C D E   **9** A B C D E

5 A B C D E   **10** A B C D E

SCORE

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. Which of the following statements is true? (there is only one)

   A. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is a convergent series.

   B. If a_n and b_n are sequences of positive numbers, if $\sum_{n=1}^{\infty} a_n$ converges and if $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} b_n$ converges.

   C. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

   D. If a_n and b_n are sequences of positive numbers, if $\sum_{n=1}^{\infty} b_n$ diverges, and $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

   E. The series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ converges.

2. The series $\sum_{n=1}^{\infty} \dfrac{(x-2)^n}{3^n}$ converges for which x?

   A. $-1 < x < 5$

   B. $-1 \leq x \leq 5$

   C. $1 \leq x \leq 5$

   D. $-1 \leq x < 5$

   E. $-1 < x \leq 5$

3. Find the partial sum S_{50} for the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right).$$

A. $\frac{1}{\sqrt{50}}-\frac{1}{\sqrt{2}}$

B. $\frac{1}{\sqrt{50}}-1$

C. $1-\frac{1}{\sqrt{51}}$

D. $1-\frac{1}{\sqrt{50}}$

E. $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{51}}$

4. Which of the following statements about the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is correct? (there is only one)

A. $\int_1^{\infty}\frac{1}{x^2}\,dx<\sum_{n=1}^{\infty}\frac{1}{n^2}$

B. $\sum_{n=1}^{\infty}\frac{1}{n^2}<\int_1^{\infty}\frac{1}{x^2}\,dx$

C. $\sum_{n=1}^{\infty}\frac{1}{n^2}<\int_2^{\infty}\frac{1}{x^2}\,dx$

D. $\sum_{n=1}^{\infty}\frac{1}{n^2}<1$

E. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ diverges

5. Consider the series $\sum_{n=1}^{\infty} \frac{5}{(1+n^2)^2}$. Applying the comparison test with the series $\sum_{n=1}^{\infty} \frac{5}{n^4}$ leads to the following conclusion.

   A. The test is inconclusive.

   B. The series converges.

   C. The series converges conditionally.

   D. The series diverges.

   E. The test cannot be applied to $a_n = \frac{5}{(1+n^2)^2}$ and $b_n = \frac{5}{n^4}$.

6. The sum of the infinite geometric series $\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$ is:

   A. $10/7$

   B. $7/10$

   C. $49/10$

   D. $49/3$

   E. $10/3$

7. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$

   A. converges conditionally

   B. converges absolutely

   C. diverges

   D. converges to 1

   E. converges to 0

8. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

A. converges conditionally but not absolutely by the ratio test

B. converges absolutely by the ratio test

C. is indeterminate for the ratio test

D. diverges by the integral test

E. diverges by comparison with $b_n = 1/n$

9. The series

$$\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$$

A. diverges by the root test

B. is indeterminate for the root test

C. converges by the root test

D. diverges by the ratio test

E. is indeterminate for the ratio test

10. The power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$$

A. has radius of convergence 0

B. converges on $(-\infty, \infty)$

C. has radius of convergence 1 and interval of convergence $(-1, 1)$

D. has radius of convergence 1 and interval of convergence $[-1, 1)$

E. has radius of convergence 1 and interval of convergence $[-1, 1]$

Free Response Questions

11. Determine whether each of the following series converges or diverges. Be sure to state which test you are using and be sure to state your reasoning clearly.

(a) (3 points) $\sum_{n=2}^{\infty} \frac{1}{n-1}$

Solution: We use the comparison test. Let $a_n = \frac{1}{n-1}$ and $b_n = \frac{1}{n}$. Since $n - 1 < n$ for all n, it follows that $\frac{1}{n} \leq \frac{1}{n-1}$. The sum $\sum b_n$ is the harmonic series, which diverges. Since $a_n \geq b_n$ it follows that the series $\sum a_n$ also diverges.

(b) (3 points) $\sum_{n=1}^{\infty} \frac{n^2}{3n^5 + 4n + 1}$

Solution: We use the comparison test. Let $a_n = \frac{n^2}{3n^5 + 4n + 1}$, and let $b_n = \frac{n^2}{3n^5}$. Since $3n^5 + 4n + 1 > 3n^5$, it follows that $a_n \leq b_n$. Up to an overall factor of $1/3$, $\sum b_n$ is the p-series with $p = 3$, and so converges. Hence $\sum a_n$ also converges.

(c) (4 points) $\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$

Solution: We use the ratio test. Let $a_n = \frac{n5^{2n}}{10^{n+1}}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \cdot \frac{5^{2(n+1)}}{5^{2n}} \cdot \frac{10^{n+1}}{10^{n+2}} = \frac{n+1}{n} \cdot \frac{25}{10}.$$

Hence

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{5}{2} > 1$$

and the series diverges.

12. Consider the power series
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3^n}(x+2)^n.$$

(a) (5 points) Find the radius of convergence for this power series. Be sure to state clearly which test you are using.

Solution: Let $a_n = \dfrac{\sqrt{n}}{3^n}(x+2)^n$. Since
$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n+1}}{\sqrt{n}} \frac{3^n}{3^{n+1}}(x+2)$$
it follows that
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}|x+2|$$
so by the ratio test the series converges provided $|x+2| < 3$. Hence, the radius of convergence is 3.

(b) (5 points) Find the integral of convergence for this power series. Be sure to justify your answer completely.

Solution: We need to check convergence at the endpoints $x = -5$ and $x = 1$. At $x = -5$ we obtain the series
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3^n}(-3)^n = \sum_{n=1}^{\infty} (-1)^n \sqrt{n}$$
which is a series $\sum_n a_n$ where $a_n = (-1)^n \sqrt{n}$. Since $\lim_{n\to\infty} a_n \neq 0$, the series diverges.
At $x = 1$, we get the series $\sum_{n=1}^{\infty} \sqrt{n}$ which diverges also.
Hence, the interval of convergence is $(-5, 1)$.

13. Recall that

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

(a) (5 points) Find the Maclaurin series for the function $\frac{1}{1+x^2}$ and determine its radius of convergence.

Solution: Substituting $-x^2$ for x we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Since the series for $(1-x)^{-1}$ converges for $|x| < 1$, the series for $(1+x^2)^{-1}$ converges for $|x^2| < 1$, i.e., $|x| < 1$ also. So the radius of convergence is 1.

(b) (3 points) Using the result of part (a) and the formula

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt,$$

find a power series representation for $\arctan(x)$.

Solution: Using the result of part (a) we compute

$$\begin{aligned} \int_0^x \frac{1}{1+t^2}\, dt &= \int_0^x \sum_{n=0}^{\infty} \left((-1)^n t^{2n} \right) dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \end{aligned}$$

(c) (2 points) For what values of x is this series representation valid? Be sure to justify your answer.

Solution: This integration is valid for $|x| < 1$ since the series is uniformly and absolutely convergent for such x. One can also check using the ratio test that this series converges for $|x| < 1$.

14. Use power series to evaluate the following limits. Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(a) (5 points) $\lim_{x\to 0} \dfrac{e^x - 1 - x}{2x^2}$

Solution: Using the Maclaurin series for e^x we compute

$$\frac{e^x - 1 - x}{2x^2} = \frac{\frac{x^2}{2} + \frac{x^3}{6} + \cdots}{2x^2}$$
$$= \frac{1}{4} + \frac{x}{12} + \ldots$$

from which it follows that

$$\lim_{x\to 0} \frac{e^x - 1 - x}{2x^2} = \frac{1}{4}.$$

(b) (5 points) $\lim_{x\to 0} \dfrac{\sin(x) - x}{5x^3}$

Solution: Using the Maclaurin series for $\sin x$, we compute

$$\frac{\sin x - x}{5x^3} = \frac{-\frac{x^3}{6} + \frac{x^5}{120} + \ldots}{5x^3}$$
$$= -\frac{1}{30} + \frac{x^2}{600} + \ldots$$

so

$$\lim_{x\to 0} \frac{\sin x - x}{5x^3} = -\frac{1}{30}.$$

15. (a) (6 points) Use the substitution $u = \ln x$ to express the integral

$$\int_2^\infty \frac{dx}{x(\ln x)^p}$$

in terms of u. Determine for which values of p the integral converges.

Solution: If $u = \ln x$ then $du = dx/x$ so

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^\infty \frac{1}{u^p}\, du.$$

This integral converges for any $p > 1$.

(b) (4 points) Using your result from part (a), find the values of p for which the series

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$$

converges. Be sure to explain your application of the integral test.

Solution: The function

$$f(x) = \frac{1}{x(\ln x)^p}$$

is a continuous function, positive for $x \geq 2$, and decreasing because the function $g(x) = x(\ln x)^p$ is increasing provided $p \geq 0$.
According to the integral test, the series

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$$

converges if and only if the improper integral

$$\int_2^\infty \frac{1}{x(\ln x)^p}$$

converges. From part (a) we see that the integral is convergent provided $p > 1$. Hence the series also converges for $p > 1$.