Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Give the first four terms of the sequence $\{a_1, a_2, \ldots\}$ defined by

$$a_n = \frac{3 \cdot 4^n}{n!}.$$

A. $\{12, 24, 32, 32\}$

B. $\{12, 12, 32, 48\}$

C. $\{12, 48, 106, 256\}$

D. $\{6, 24, 32, 48\}$

E. $\{6, 12, 48, 32\}$

2. (5 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^3 - 2n + 1}$ converge or diverge?

A. Diverges because $\lim_{n\to\infty} \frac{2n+1}{n^3-2n+1} \neq 1$.

B. Converges by the limit comparison test to $\sum_{n=1}^{\infty} \frac{2}{n^2}$.

C. Converges because $\lim_{n\to\infty} \frac{2n+1}{n^3-2n+1} = 0$.

D. Diverges by the limit comparison test to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

E. Diverges by the limit comparison test to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. (5 points) What is the value of C if $\sum_{n=0}^{\infty}(2+C)^n = 3$?

   A. $-\frac{1}{3}$
   B. $\frac{1}{2}$
   C. $\frac{4}{5}$
   D. $-\frac{4}{3}$
   E. No such C exists.

4. (5 points) Which of the following series converge?

   A. $\sum_{n=5}^{\infty} \frac{n+2}{\sqrt{n^2-1}}$
   B. $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n+2}}$
   C. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{10}}$
   D. None of the above series converge.
   E. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n}}$

5. (5 points) What would you compare $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{3^n - n^5 + 1}$ to for a conclusive limit comparison test?

   A. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^5}$

   B. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{3}{2}\right)^n$

   C. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n$

   D. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

   E. The limit comparison test can't be used to understand convergence for this series.

6. (5 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{4^n + n}$ converge or diverge?

   A. Converges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{2^{n+1}}{4^{n+1} + n + 1} \frac{4^n + n}{2^n} = \frac{1}{2} < 1$**.**

   B. Diverges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{2^{n+1}}{4^{n+1} + n + 1} \frac{4^n + n}{2^n} = \frac{1}{2} > 0$.

   C. Diverges by the limit comparison test because $\displaystyle\lim_{n\to\infty} \frac{2^n}{4^n + n + 1} = 0$.

   D. Converges by the divergence test because $\displaystyle\lim_{n\to\infty} \frac{2^n}{4^n + n + 1} = \frac{2}{4} = \frac{1}{2} \neq 0$.

   E. Diverges by the ratio test because $\displaystyle\lim_{n\to\infty} \frac{4^{n+1} + n + 1}{2^{n+1}} \frac{2^n}{4^n + n} = 2 > 1$.

7. (5 points) Find the smallest value of N so that S_N approximates $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ to within an error of at most .0001.

   A. $N = 49$
   B. $N = 149$
   C. $N = 99$
   D. $N = 24$
   E. $N = 9$

8. (5 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{4^n n^2}$?

   A. $(-\frac{4}{3}, \frac{4}{3}]$
   B. $[-1, \frac{5}{3}]$
   C. $(-1, \frac{5}{3})$
   D. $[-1, 1]$
   E. $(-\infty, \frac{5}{3}]$

9. (5 points) Which power series represents the function $\dfrac{x^2}{e^{2x}}$ on the interval $(-\infty, \infty)$?

A. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} x^{2n}$

B. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n+1)!} x^{4n+2}$

C. $\displaystyle\sum_{n=0}^{\infty} (-2)^n x^{2n}$

D. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

E. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^{n+2}$

10. (5 points) Use the fact that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$ to find the sum of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(n+1)}.$$

Be careful with the index!

A. $-\frac{1}{2}$

B. $\frac{3}{2}$

C. $\frac{2}{3}$

D. $\frac{1}{2}$

E. $\frac{5}{2}$

Free Response Questions

11. (a) (5 points) Decide if the series converges or diverges (Clearly state which test(s) are used):

$$\sum_{n=2}^{\infty} \frac{3n^2}{\sqrt{n^7 - 2n - 1}}.$$

Solution: Converges by a limit comparison test with the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

(b) (5 points) Decide if the series converges or diverges (Clearly state which test(s) are used):

$$\sum_{n=2}^{\infty} \frac{1}{2^n + n}$$

Solution: Converges by comparison test with the geometric series $\sum_{n=2}^{\infty} \frac{1}{2^n}$.

12. (10 points) Verify that the integral test applies to the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$, and use it to decide whether or not the series converges.

Solution: The corresponding improper integral is $\int_1^\infty \frac{\ln(x)}{x^2} dx$. Integration by parts with $u = \ln(x)$ and $dv = x^{-2}$ gives $du = x^{-1}$ and $v = -x^{-1}$, so we obtain:

$$\int \frac{\ln(x)}{x^2} dx = -\frac{\ln(x)}{x} + \int x^{-2} dx = -x^{-1}(\ln(x) + 1)$$

The improper integral is then computed by the limit $\lim_{t\to\infty} 1 - \frac{\ln(t)+1}{t} = 1$

13. Are the series below **absolutely convergent, conditionally convergent,** or **divergent**? Justify your answer.

(a) (6 points)

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n)}{n}$$

Solution: The series converges conditionally by the alternating series test:

$\lim_{n\to\infty} \frac{\ln(n)}{n}$ = (L'Hopital's rule) $\lim_{n\to\infty} \frac{1}{n} = 0$.
Let $f(x) = x^{-1}\ln(x)$, then $f'(x) = -x^{-2}\ln(x) + x^{-2} = x^{-2}(1 - \ln(x)$; this is negative for large values of x, so the series is eventually decreasing.

However, the absolute value series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges by comparison test with the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$.

(b) (4 points)

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{(n+1)!}$$

Solution: The series converges absolutely by the ratio test.

14. (a) (4 points) What is the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{2^n}{(5n)^n} x^n$?

Solution: By the root test or ratio test, the interval is $(-\infty, \infty)$.

(b) (6 points) Find the **center** and **radius** of convergence for the following power series (**note: you do not need to check the endpoints!**):

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n (x+1)^n$$

Solution: This will converge by the root test or ratio test when $|\frac{2}{3}(x+1)| < 1$, or equivalently $|x+1| < \frac{3}{2}$. Rewriting as a system of inequalities, we get $-\frac{3}{2} < x+1 < \frac{3}{2}$, so that $-\frac{5}{2} < x < \frac{1}{2}$. The center of this interval is -1 and the radius is $\frac{3}{2}$.

15. (a) (5 points) Write a series expansion for the function $f(x) = \dfrac{1}{(1-x)^2}$ centered at $x = 0$. (**Hint: first find the antiderivative of $f(x)$.**)

> **Solution:** The function $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$, so we obtain the answer by taking the derivative of the series $\sum_{n=0}^{\infty} x^n$, giving $\sum_{n=1}^{\infty} nx^{n-1}$ or $\sum_{n=0}^{\infty}(n+1)x^n$.

(b) (5 points) Find the first **five** coefficients of the Taylor series for $\sin(x)$ centered at $a = \frac{3\pi}{4}$. **You must find and label the value of each of the five coefficients separately.**

> **Solution:** $\sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$, $\cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$, $-\sin(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$, $-\cos(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$, $\sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$.
> $c_0 = \frac{1}{\sqrt{2}}$, $c_1 == -\frac{1}{\sqrt{2}}$, $c_2 = -\frac{1}{\sqrt{2}}\frac{1}{2}$, $c_3 = \frac{1}{\sqrt{2}}\frac{1}{6}$, $c_4 = \frac{1}{\sqrt{2}}\frac{1}{24}$.