Exam 3

19 November 2013

Name: ______________________________

Section: ____________ Instructor or TA: ____________________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 4 multiple choice questions and 8 free response questions. Record your answers to the multiple choice questions below on this page by filling in the single circle corresponding to the correct answer. All other work must be done in the body of the exam.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **3** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **4** (A) (B) (C) (D) (E)

SCORE

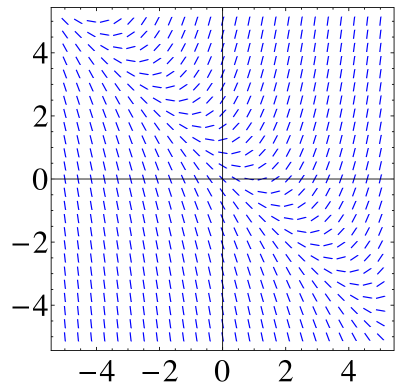
Multiple Choice	5	6	7	8	9	10	11	12	Total Score
20	10	10	10	10	10	10	10	10	100

Multiple Choice Questions

1. Which one of the following differential equations corresponds to the given slope field?

   A. $y' = 2 - y$ B. $y' = x + y - 1$ C. $y' = x(2 - y)$ D. $y' = y\left(1 - \frac{y}{5}\right)$ E. None of these.

B



2. Use Euler's Method with a step size of $h = 0.5$ to estimate the value of $y(1)$, where y is the solution of the initial value problem:

$$y' = x + y \qquad \text{and} \qquad y(0) = 1$$

E

A. 1 B. $\frac{3}{2}$ C. $\frac{1+\sqrt{5}}{2}$ D. 2 E. $\frac{5}{2}$

$y(0.5) \approx 1 + 0.5(0+1) = 1.5$

$y(1) \approx 1.5 + 0.5(0.5 + 1.5)$

$= 1.5 + 1 = 2.5$

3. Determine whether the following series converge or diverge.

   A. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{n!}$ ABS

   B. $\displaystyle\sum_{n=1}^{\infty} n^4\, 3^n$ DIV

   A. A and B both converge
   B. A converges conditionally, B diverges
   C. A diverges, B converges
   D. A converges absolutely, B diverges
   E. A and B both diverge

D

4. What is the limit of the sequence $a_n = \dfrac{3\sin(n)}{2 + \ln(n)}$?

   A. $\frac{3}{2}$
   B. 0
   C. 3
   D. The sequence diverges without bound.
   E. The sequence is bounded and divergent.

B

Free Response Questions

You must show all of your work in these problems to receive credit. Answers without corroborating work will receive no credit.

5. A state game commission releases 40 elk into a game refuge. Assume the elk population, P, grows according the following logistic model with a growth constant of $k = \ln(11/9)$ per year:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{4000}\right)$$

(a) What is the carrying capacity?

$A = 4000$

(b) The general solution to a logistic differential equation is $\frac{A}{1 - e^{-kt}/C}$, where A is the carrying capacity. Use the given information to find C.

$40 = \frac{4000}{1 - e^{-\ln(11/9)0}/C} = \frac{4000}{1 - (9/11)^0/C}$

$1 - 1/c = 100$ $\quad \boxed{C = -1/99}$

(c) Use the result from part (~~a~~ b) to estimate the elk population after 15 years.

$y(15) \approx \frac{4000}{1 + 99 \cdot (9/11)^{15}} \approx \boxed{680}$

(d) At what time t is the population of elk growing the fastest?
HINT: First use the given differential equation to find a population for which the **growth rate** is highest. Then solve for t.

$\frac{dP}{dt}$ highest $P(1 - \frac{P}{4000})$ has a max,

@ $P = 2000$.

Solve $2000 = \frac{4000}{1 + 99(9/11)^t}$

$1 + 99(9/11)^t = 2$

$(9/11)^t = \frac{1}{99}$

$t = \frac{\ln(1/99)}{\ln(9/11)}$

$\approx \boxed{22.9 \text{ years}}$

6. Let $f(x) = x^3 e^x$.

   (a) Write down the Taylor series for $f(x)$ centered at $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x^3 e^x = x^3 + x^4 + \frac{x^5}{2} + \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!}$$

$$\underline{\text{or}} \quad \sum_{n=3}^{\infty} \frac{x^n}{(n-3)!}$$

   (b) Find the radius of convergence for this series.

Ratio test

$$\lim_n \frac{|x|^{n+4}/(n+1)!}{|x|^{n+3}/n!} = \lim_n \frac{|x|}{n+1} = 0 \quad \text{for } \underline{\text{any}}\ x$$

$$R = \infty$$

   (c) Use your answer in (a) to find $f^{(6)}(0)$.

$$\frac{1}{6} = a_6 = \frac{f^{(6)}(0)}{6!}$$

$$\text{so} \quad f^{(6)}(0) = \frac{6!}{6} = 5! = 120$$

7. (a) Find a series expression for the function $g(x) = \dfrac{1}{1+2x^2}$.

$$g(x) = \frac{1}{1-(-2x^2)} = 1 - 2x^2 + 4x^4 - \cdots$$
$$= \sum_{n=0}^{\infty} (-2)^n x^{2n}$$

(b) What are the radius and interval of convergence for this series?

Since $\sum_n x^n$ converges $\Leftrightarrow |x| < 1$,

it follows that $\sum_n (-2x^2)^n$

converges $\Leftrightarrow |-2x^2| < 1$,

so $|x|^2 < \frac{1}{2}$

$|x| < \boxed{\frac{1}{\sqrt{2}} = R}$

interval is $\boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

OR

Root test:

$\lim_n \sqrt[n]{(2|x|^2)^n} = \lim_n 2|x|^2 = 2|x|^2$

So by Root test, conv. if

$2|x|^2 < 1$, or $|x| < \boxed{\frac{1}{\sqrt{2}} = R}$

endpoints

$x = 1/\sqrt{2}$ $\quad \sum (-2)^n \cdot \frac{1}{2^n} = \sum (-1)^n$

$x = -1/2$ $\quad \sum (-2)^n \cdot \frac{1}{2^n}$ // Diverges by Div Test

So interval is $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

8. Use the *method of integrating factors* to solve the following initial value problem.

$$xy' = y - x \qquad y(1) = 2$$

$y' - \frac{1}{x}y = 1 \qquad y(1) = 2$

$A(x) = -1/x \qquad B(x) = -1$

$\alpha(x) = e^{-\int \frac{dx}{x}} = e^{-\ln(x)} = \frac{1}{x}$

$y(x) = \frac{1}{\alpha(x)} \int \alpha(x) B(x)\, dx$

$= x \int \frac{-1}{x} dx = x(-\ln|x| + C)$

$2 = 1(-\ln 1 + C) = C$

so $\boxed{y(x) = x(-\ln|x| + 2)}$

9. Suppose $\{a_n\}$ is a sequence with $a_n = \dfrac{1}{n(n-1)} = \dfrac{1}{n} - \dfrac{1}{n-1}$.

(a) Determine if the series $S = \sum_{n=2}^{\infty} a_n$ is convergent. Please show the details of your work.

Use Limit Comp Test with $\sum \frac{1}{n^2}$

$$\lim_n \frac{\frac{1}{n(n-1)}}{1/n^2} = \lim \frac{n^2}{n^2-n} = \lim \frac{1}{1-1/n} = 1 > 0$$

Since $\sum \frac{1}{n^2}$ converges (p-series w/ $p=2$), $\sum \frac{1}{n(n-1)}$ also converges

(b) Find the value of $S_6 = \sum_{n=2}^{6} a_n$.

$$S_6 = \underbrace{\left(1 - \cancel{\tfrac{1}{2}}\right)}_{n=2} + \underbrace{\left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right)}_{n=3} + \underbrace{\left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right)}_{n=4} + \underbrace{\left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{5}}\right)}_{n=5} + \underbrace{\left(\cancel{\tfrac{1}{5}} - \tfrac{1}{6}\right)}_{n=6} = 1 - \frac{1}{6} = 5/6$$

(c) Find the value of S.

$$S = \lim_n S_n = \lim_n 1 - \frac{1}{n} = \underline{1}$$

10. Determine whether $\sum_{n=1}^{\infty} \dfrac{n^2+1}{\sqrt{n^7+n^3}}$ converges or diverges. Please show the details of your work.

Use Limit Comp Test with $\sum \frac{1}{n^{3/2}}$

$$\lim_n \frac{\frac{n^2+1}{\sqrt{n^7+n^3}}}{1/n^{3/2}} = \lim_n \frac{n^{7/2}+n^{3/2}}{\sqrt{n^7+n^3}} = \lim_n \sqrt{\frac{(n^{7/2}+n^{3/2})^2}{n^7+n^3}}$$

$$= \lim \sqrt{\frac{n^7+2n^5+n^3}{n^7+n^3}} = \underline{1} > 0$$

since $\sum \frac{1}{n^{3/2}}$ converges ($p = 3/2 > 1$), original series converges too

11. Given the series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+19}$.

(a) Determine if S is absolutely convergent, conditionally convergent or divergent. Please show the details of your work.

Limit Comp with $\frac{1}{n^2}$

$$\lim_n \frac{\frac{1}{n^2+19}}{1/n^2} = \lim_n \frac{n^2}{n^2+19} = \lim \frac{1}{1+\frac{19}{n^2}} = \underline{1} > 0$$

Since $\sum \frac{1}{n^2}$ conv. $(p = 2 > 1)$, it follows that $\sum \frac{(-1)^n}{n^2+19}$ converges a<u>bsolutely</u>

(b) Given the fact that $|S - S_N| \le a_{(N+1)}$, where $S = \sum_{n=1}^{\infty}(-1)^n a_n$ and S_N is the Nth partial sum of S. For this problem, what is the smallest N such that $|S - S_N| < 10^{-2}$?

Want $|S-S_N| \le a_{N+1} < \frac{1}{100}$

$$\frac{1}{(N+1)^2+19} = \frac{1}{N^2+2N+20} < \frac{1}{100} \iff 100 < N^2+2N+20$$

$$80 < N^2+2N$$

$$8^2+2\cdot 8 = 80 \not> 80$$

$$9^2+2\cdot 9 = 99 > 80$$

$$\boxed{N=9}$$

12. Find the radius and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

Ratio Test: $\lim_n \frac{\frac{|x|^{n+1}}{(n+1)3^{n+1}}}{\frac{|x|^n}{n3^n}} = \lim_n \frac{|x|}{3}\frac{n}{n+1} = \frac{|x|}{3} < 1$

if $|x| < 3$ so $\boxed{R=3}$

endpoints $x=3$ $\sum \frac{3^n}{n3^n} = \sum \frac{1}{n}$ diverges (p-test, $p = 1 \not> 1$)

$x=-3$ $\sum \frac{(-3)^n}{n3^n} = \sum \frac{(-1)^n}{n}$ converges (alternating harmonic)

so the interval of convergence is $\boxed{[-3, 3)}$