MA 114 Exam 3 Fall 2018

Name: Solutions Section: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1	A	~~B~~	C	D	E	**6**	A	B	C	~~D~~	E
2	A	~~B~~	C	D	E	**7**	A	B	C	D	~~E~~
3	A	B	~~C~~	D	E	**8**	A	B	C	D	~~E~~
4	~~A~~	B	C	D	E	**9**	A	B	C	~~D~~	E
5	A	B	~~C~~	D	E	**10**	A	~~B~~	C	D	E

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Find the average value of the function $f(x) = \sin(x)$ on the interval $[\pi/2, \pi]$.

   A. $-2/\pi$
   B. $2/\pi$
   C. 0
   D. $-1/\pi$
   E. $1/\pi$

$$\frac{1}{\pi/2}\int_{\pi/2}^{\pi} \sin x \, dx$$

$$= \frac{2}{\pi}\Big[-\cos x\Big]_{x=\pi/2}^{x=\pi} = \frac{2}{\pi}\left[0-(-1)\right] = \frac{2}{\pi}$$

2. (5 points) The base of a solid is the square $S = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 2\}$. Suppose that the cross sections of S by a plane perpendicular to the y-axis and containing the line $y = a$ are rectangles with height $3a$. Find the volume of the solid.

   A. 9
   B. 12
   C. 18
   D. 24
   E. 36

$$\int_0^2 (3y)\cdot 2\, dy = \Big[3y^2\Big]_{y=0}^{2} = 12$$

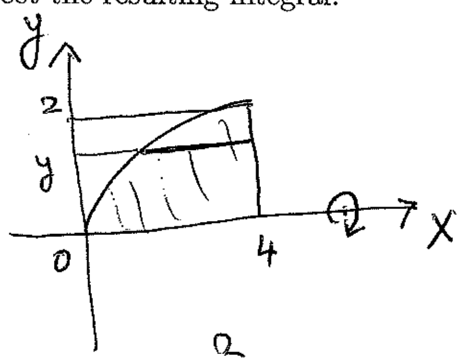
3. (5 points) We create a solid ball B by rotating the region between the curves $y = \sqrt{25 - x^2}$ and the x-axis about the x-axis. We slice B with a plane perpendicular to the x-axis and containing the line with equation $x = 4$. Find the area of the resulting cross-section.

   A. π
   B. 4π
   C. 9π
   D. 16π
   E. 25π

The cross-section is a disk of radius $\sqrt{25-4^2} = 3$. Thus the area is $\pi \cdot 3^2 = 9\pi$.

4. (5 points) Let R be the region bounded by $x = y^2$, $y = 0$ and $x = 4$. The region R is rotated about the x-axis to form a solid of revolution S. Use the method of cylindrical shells to find the volume of S. Select the resulting integral.

   A. $2\pi \int_0^2 y(4 - y^2)\, dy$
   B. $2\pi \int_0^2 y(2 - y^2)\, dy$
   C. $\pi \int_0^2 4 - y^4\, dy$
   D. $\pi \int_0^4 16 - y^4\, dy$
   E. $2\pi \int_0^4 y^2(4 - y)\, dy$



circum. $2\pi y$
height. $4-y^2$

$$Vol(S) = \int_0^2 (2\pi y)(4-y^2)\, dy$$

5. (5 points) Find the length of the line segment with endpoints $(1, 2)$ and $(5, 5)$.

   A. 3
   B. 4
   C. 5
   D. 7
   E. None of the other answers are correct.

$$\sqrt{(5-1)^2 + (5-2)^2} = \sqrt{25} = 5$$

6. (5 points) Three masses are located in the plane. The first mass is 2 grams at $(0,0)$, the second mass is 4 grams located at $(3,0)$ and the third mass is 2 grams located at $(3,4)$. Find the center of mass of the system.

A. $(3/2,2)$
B. $(6,8/3)$
C. $(1,7/4)$
D. $(9/4,1)$
E. $(8/3,6)$

$m = 2+4+2=8$

$\bar{x} = \frac{1}{8}(2\cdot 0 + 4\cdot 3 + 2\cdot 3) = \frac{18}{8} = \frac{9}{4}$

$\bar{y} = \frac{1}{8}(2\cdot 0 + 4\cdot 0 + 2\cdot 4) = \frac{8}{8} = 1$

7. (5 points) Consider the curve with parametric equations $x(t) = t^2$, $y(t) = t^3 - t + 1$. Find the slope of the tangent line to the curve at $(4,7)$.

A. $3/4$
B. $5/4$
C. $7/4$
D. $9/4$
E. $11/4$

When $(t^2, t^3-t+1) = (4,7)$, we have $t = \pm 2$, $t^3 - t + 1 = \pm 6 + 1$. So $\boxed{t=2}$.

Then the slope of the tangent is

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-1}{2t}$.

For $t=2$, the slope is $\frac{3\cdot 2^2 - 1}{2\cdot 2} = \boxed{11/4}$

8. (5 points) Find parametric equations which describe the circle centered at $(2, 1)$ with radius 3.

   A. $x(t) = 1 + 3\cos(t), y(t) = 3 + 3\sin(t),\ 0 \le t < 2\pi$

   B. $x(t) = 1 + 3\sin(t), y(t) = 3 + 3\cos(t),\ 0 \le t < 2\pi$

   C. $x(t) = 3 + 2\sin(t), y(t) = 3 + \cos(t),\ 0 \le t < 2\pi$

   ~~D.~~ $x(t) = 1 + 3\sin(t), y(t) = 2 + 3\cos(t),\ 0 \le t < \pi$

   E. $x(t) = 2 + 3\cos(t), y(t) = 1 + 3\sin(t),\ 0 \le t < 2\pi$

Since the center is at $(2,1)$, the correct equation is E.

9. (5 points) Let C be the line segment with parametric equations $x(t) = 3t$, $y(t) = 4t$, for $3 \le t \le 5$. Find the surface area obtained if we rotate C about the x-axis.

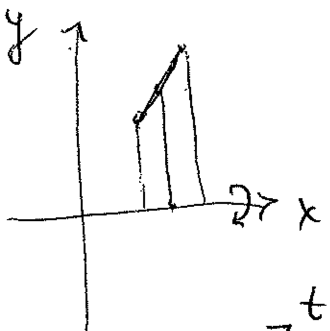
   A. 240π

   B. 260π

   C. 280π

   D. 320π

   E. 340π



$$S = \int_3^5 [2\pi(4t)]\, ds = \int_3^5 40\pi t\, dt$$

$$ds = \sqrt{3^2+4^2}\, dt = 5\, dt$$

Thus $S = 40\pi \dfrac{t^2}{2}\Big]_{t=3}^{t=5} = 20\pi(5^2 - 3^2) = \boxed{320\pi}$

10. (5 points) Find the value of a so that the curve with parametric equations $x(t) = 2t - 1$, $y(t) = at^2 - t$ contains the point $(1, 1)$.

   A. 1

   B. 2

   C. 3

   D. 4

   E. 5

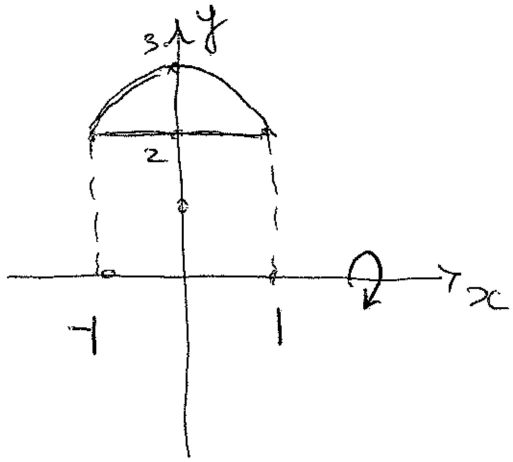
From $2t - 1 = 1$, we get $t = 1$

Then $at^2 - t = a - 1 = 1$ and $a = 2$

Free Response Questions

11. Let R be the region enclosed by $y = 3 - x^2$ and $y = 2$. The region R is rotated about the x-axis to obtain a solid of revolution, S.

   (a) (5 points) Write an integral to give the volume of S. Indicate clearly which method you are using.



$$\mathrm{Vol}(S) = \int_{-1}^{1} \pi \left[(3-x^2)^2 - 2^2 \right] dx$$

   (b) (5 points) Evaluate the integral and give the volume of S.

$$\mathrm{Vol}(S) = \pi \int_{-1}^{1} \left[5 - 6x^2 + x^4 \right] dx$$

$$= 2\pi \left[5x - 2x^3 + \frac{x^5}{5} \right]_{x=0}^{x=1} = \boxed{\frac{32\pi}{5}}$$

12. Consider a lamina H which is the semi-circle enclosed by $x^2 + y^2 = 4$ and the x-axis and lies above the x-axis. Assume the density of the lamina is 5 units of mass per unit of area.

(a) (7 points) Find the total mass M and the moments M_x and M_y of the lamina H. Hint: You may use geometry to evaluate the integral for the mass.

$$M = 5 \cdot \text{Area}(H) = \frac{5}{2}(\pi \cdot 2^2) = 10\pi$$

$M_y = 0$ by symmetry

$$M_x = \frac{5}{2}\int_{-2}^{2}(4-x^2)\,dx = 5\left[4x - \frac{x^3}{3}\right]_{x=0}^{2}$$

$$= 5\left[8 - \frac{8}{3}\right] = \frac{80}{3}.$$

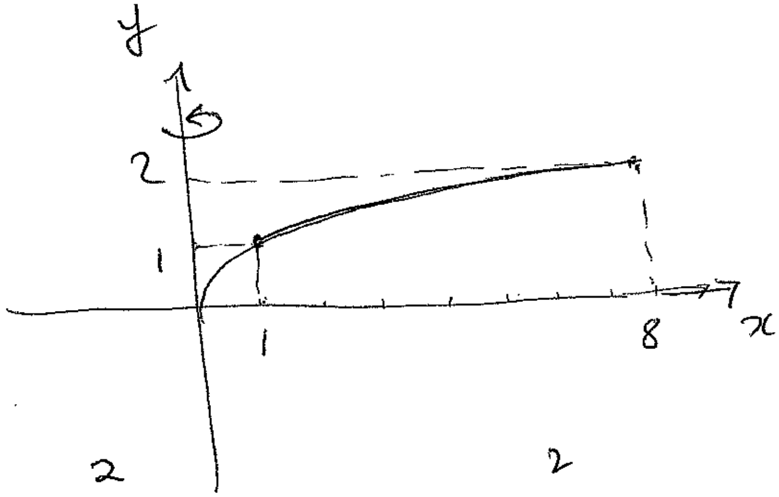
(b) (3 points) Find the center of mass for H.

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(0, \frac{8}{3\pi}\right)$$

since $\frac{80/3}{10\pi} = \frac{8}{3\pi}$

13. Consider the curve C given $x = y^3$ for $1 \leq y \leq 2$.

(a) (5 points) The curve C is rotated about the y-axis to obtain a surface of revolution S. Express the area of S as an integral.



$$\text{Area}(S) = \int_1^2 (2\pi x\, ds) = 2\pi \int_1^2 y^3 \sqrt{1+9y^4}\, dy$$

(b) (5 points) Evaluate the integral to find the area of S.

$u = 1+9y^4$
$du = 36y^3 dy$

$y=1 \Rightarrow u=10$
$y=2 \Rightarrow u=145$

$$\text{Area}(S) = \frac{2\pi}{36} \int_{10}^{145} \sqrt{u}\, du$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} \left[u^{3/2}\right]_{u=10}^{u=145}$$

$$= \frac{\pi}{27} \left(145\sqrt{145} - 10\sqrt{10}\right)$$

14. Consider the curve C with parametric equations $x(t) = 2\sqrt{1-t^2}$, $y(t) = 2t$, $0 \leq t \leq 1/2$.

(a) (5 points) Express the length of curve C as an integral.

$x = 2\sqrt{1-t^2}$ $\quad x' = \dfrac{-2t}{\sqrt{1-t^2}}$ $\quad y = 2t$ $\quad y' = 2$

$ds = \sqrt{\dfrac{4t^2}{1-t^2} + 4}\, dt = 2\sqrt{\dfrac{1}{1-t^2}}\, dt = \dfrac{2}{\sqrt{1-t^2}}\, dt.$

So $L = \displaystyle\int_0^{1/2} \frac{2\,dt}{\sqrt{1-t^2}}.$

(b) (5 points) Evaluate the integral in part a) to find the length of the curve C. Hint: The curve is part of a circle, so you may check your answer by finding the length without calculus. However, to receive credit you must use calculus.

$L = \displaystyle\int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{\cos\theta}$

$= 2 \cdot \dfrac{\pi}{6} = \dfrac{\pi}{3}$

$t = \sin\theta$

$dt = \cos\theta\, d\theta$

$0 \leq \theta \leq \frac{\pi}{6}$

$\sqrt{1-t^2} = \cos\theta$

15. Consider the curve C with parametric equations $x(t) = 2t - 4$ and $y(t) = t^2 - t$.

(a) (5 points) Find the tangent line to the curve at the point $(x, y) = (2, 6)$. Put your answer in slope-intercept form $y = mx + b$.

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-1}{2}$$

$(2t-4, t^2-t) = (2,6)$ for $t=3$. Then $m = \frac{5}{2}$

and $y - 6 = \frac{5}{2}(x-2)$, or

$$\boxed{y = \frac{5}{2}x + 1}$$

(b) (5 points) Find the point(s) on the curve C where the curve has a horizontal tangent line.

$m = 0$ when $2t - 1 = 0$, i.e. $t = \frac{1}{2}$

and $(x, y) = (-3, -1/4)$.