MA 114 — Calculus II Spring 2014
Exam 3 April 15, 2014

Name: Key

Section: ____________

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:** Show all your work on the page of the problem. Show all your work. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	~~B~~	C	D	E
2	~~A~~	B	C	D	E
3	A	B	C	~~D~~	E
4	A	B	~~C~~	D	E

Exam Scores

Question	Score	Total
MC		20
5		14
6		15
7		18
8		15
9		18
Total		100

Unsupported answers for the free response questions may not receive credit!

Hint: Recall that the general solution of the first-order linear differential equation

$$y' + A(x)y = B(x)$$

is $y(x) = \dfrac{1}{\alpha(x)}\left[\int \alpha(x)B(x)\,dx + C\right]$, where $\alpha(x) = e^{\int A(x)\,dx}$ and C is a constant.

Record the correct answer to the following problems on the front page of this exam.

1. Which of the following options is true for the differential equation

$$x \cdot y' = (y-1)^3 \cdot \left(1 - \frac{y}{5}\right)?$$

A. It has no constant solutions.

B. $y = 1$ and $y = 5$ are solutions.

C. $y = 1$ is a solution, but $y = 5$ is not a solution.

D. $y = 5$ is a solution, but $y = 1$ is not a solution.

E. None of the above.

2. Which of the following options is true for the sequence $\left\{ \dfrac{1}{\ln(n)} \right\}_{n=2}^{\infty}$?

A. It is decreasing and convergent.

B. It is decreasing and divergent.

C. It is increasing and convergent.

D. It is increasing and divergent.

E. None of the above.

Record the correct answer to the following problems on the front page of this exam.

3. Which of the following options is true for the infinite series $\sum_{n=1}^{\infty} \cos(\frac{1}{n})$?

A. It absolutely convergent and convergent.

B. It absolutely convergent, but divergent.

C. It is convergent, but not absolutely convergent.

~~D.~~ It is divergent.

E. None of the above.

4. Which of the following options is true for the infinite series $\sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{n}$?

A. It absolutely convergent and convergent.

B. It absolutely convergent, but divergent.

~~C.~~ It is convergent, but not absolutely convergent.

D. It is divergent.

E. None of the above.

Free Response Questions: Show your work!

5. (14 points) A bacteria culture is grown starting with 160 bacteria. After 4 days there are 1000 bacteria. Assume the bacteria population grows according to the model $y'(t) = ky\left(1 - \frac{y}{8000}\right)$, where $y(t)$ is the number of bacteria after t days and k is a constant. (Recall that the general solution to the given logistic equation is $y(t) = \frac{8000}{1 + De^{-kt}}$, where D is a constant.)

(a) (8 points) Determine the constant k. Give the **exact** answer.

We are given $\underbrace{y(0) = 160}_{1}$. Thus, $160 = \frac{8000}{1+D}$, so

$1 + D = \frac{8000}{160} = 50$, hence $D = 49$. 2

Using $\underbrace{y(4) = 1000}_{1}$, we get $1000 = \frac{8000}{1+49e^{-k4}}$, so

$$1 + 49e^{-k4} = \frac{8000}{1000} = 8,$$

$$e^{4k} = 7$$

$$\underline{\underline{k = \frac{\ln 7}{4}}}.$$

3

(b) (6 points) Estimate when the population exceeds 4000 bacteria.

We solve $4000 = y(t) = \frac{8000}{1+49e^{-kt}}$ for t: 2

$$1 + 49e^{-kt} = \frac{8000}{4000} = 2$$

$$e^{kt} = 49$$

$$t = \frac{\ln 49}{k} \overset{(a)}{=} \frac{4 \cdot \ln 49}{\ln 7} = \underline{\underline{8}}.$$

4

After 8 days there will be 4000 bacteria.

Free Response Questions: Show your work!

6. (15 points) Find the solution to the initial value problem $xy' - 2y = 5x^2 + x$, where $x \geq 1$ and $y(1) = 2$.

Dividing by x, we get $y' \underbrace{- \frac{2}{x}}_{A(x)} y = \underbrace{5x+1}_{B(x)}$. ①

We compute $\int A(x)\,dx = \int -\frac{2}{x}\,dx = -2 \ln x$. $(x \geq 1)$ ②

Hence $\alpha(x) = e^{\int A(x)dx} = e^{-2\ln x} = (e^{\ln x})^{-2} = x^{-2}$. ③

$$\int \alpha(x) B(x)\,dx = \int x^{-2}[5x+1]\,dx = \int [\tfrac{5}{x} + x^{-2}]\,dx = 5\ln x - \tfrac{1}{x} + C.$$ ④

Thus, the general solution is

$$y(x) = x^2[5\ln x - \tfrac{1}{x} + C] = 5x^2 \ln x - x + Cx^2.$$ ②

The initial condition gives

$$2 = y(1) = -1 + C, \text{ so } C = 3.$$

Hence $y(x) = \underline{\underline{5x^2 \ln x - x + 3x^2}}$. ③

Free Response Questions: Show your work!

7. (18 points) Determine whether the following series converge or diverge. Please show the details of your argument.

(a) (7 points) $\sum_{n=1}^{\infty} \underbrace{\frac{2n}{7n^3-4}}_{a_n}$. (Hint: You may want to use the limit comparison test.)

Use $b_n = \frac{2}{7}\frac{1}{n^2}$. Then

$$\frac{a_n}{b_n} = \frac{2n}{7n^3-4}\,\frac{7n^2}{2} = \frac{14n^3}{14n^3-8} \xrightarrow[n\to\infty]{} \frac{14}{14} = 1.$$ (5)

Since $\sum_{n=1}^{\infty} b_n = \frac{2}{7}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series with $p=2$), (1)

the limit comparison test gives that also $\sum_{n=1}^{\infty} a_n$ converges. (1)

(b) (6 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This is an alternating series with $\underbrace{a_n = \frac{1}{\sqrt{n}} > 0}_{(1)}$.

Note $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. (1)

Moreover, the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing because $\sqrt{x}$ is increasing, so $\frac{1}{\sqrt{x}}$ is a decreasing function. (2)

Hence, the given series converges by the Leibniz test. (1)

or mentioning that series is alternating

(1) for addressing all three conditions

(c) (5 points) $\sum_{n=1}^{\infty} \underbrace{\frac{2^{1+6n}}{(2n)^n}}_{a_n}$.

$$\sqrt[n]{|a_n|} = \frac{\left(2^{1+6n}\right)^{\frac{1}{n}}}{2n} = \frac{2^{\frac{1}{n}+6}}{2n} \xrightarrow[n\to\infty]{} 0 \text{ because}$$ (4)

$\underbrace{\lim_{n\to\infty} 2^{\frac{1}{n}+6} = 2^6}_{(1)}$. Hence, the given series converges by the root test.

Free Response Questions: Show your work!

8. (15 points) Determine the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \underbrace{\frac{4^n}{n}(x-5)^n}_{a_n}.$$

Please, be sure to discuss the endpoints of the interval.

Use the ratio test to find the radius of convergence:

4 $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{4^{n+1}(x-5)^{n+1}}{n+1} \cdot \frac{n}{4^n(x-5)^n}\right| = 4|x-5|\frac{n}{n+1} \xrightarrow[n\to\infty]{} 4|x-5|.$

4 Since $4|x-5|<1$ if and only if $|x-5|<\frac{1}{4}$, the radius of convergence is $\underline{R=\frac{1}{4}}$.

Hence, the endpoints of the interval of convergence are $5-\frac{1}{4}$ and $5+\frac{1}{4}$.

3 For $x = 5-\frac{1}{4} = \frac{19}{4}$, we get

$$\sum_{n=1}^{\infty} \frac{4^n}{n}\left(5-\frac{1}{4}-5\right)^n = \sum_{n=1}^{\infty} \frac{4^n}{n}\left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges because the alternating harmonic series converges.

3 For $x = 5+\frac{1}{4} = \frac{21}{4}$, we get

$$\sum_{n=1}^{\infty} \frac{4^n}{n}\left(5+\frac{1}{4}-5\right)^n = \sum_{n=1}^{\infty} \frac{1}{n},$$ the harmonic

series. It is divergent. Hence, the interval of convergence is

1 $$\underline{\underline{\left[\frac{19}{4}, \frac{21}{4}\right)}}.$$

Free Response Questions: Show your work!

9. (18 points)

(a) (14 points) Find the Taylor series centered at zero of the function $\ln(x+5)$.

(6) Observe that $\ln(x+5) = \int \frac{1}{x+5}\,dx$ if $x+5>0$.

$$\frac{1}{x+5} = \frac{1}{5}\frac{1}{1+\frac{x}{5}} = \frac{1}{5}\frac{1}{1-(-\frac{x}{5})} \underset{\text{series}}{\overset{\text{geom.}}{=}} \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}x^n \text{ for } \left|\frac{x}{5}\right|<1.$$

(6) Integrating term-by-term, this gives

$$\ln(x+5) = \sum_{n=0}^{\infty}\left[\int \frac{(-1)^n}{5^{n+1}}x^n\,dx\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)5^{n+1}}x^{n+1} + A.$$

(1) For $x=0$, we get $\ln(5) = A$, so

(1)

$$\ln(x+5) = \ln 5 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)5^{n+1}}x^{n+1} \quad \text{or}$$

$$\ln(x+5) = \ln 5 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot 5^n}x^n.$$

(b) (4 points) Find the Taylor series centered at zero of the function $x^3\ln(x^2+5)$. (Hint: You may want to apply your answer from part (a).)

Using (a), we get

$$x^3\ln(x^2+5) = x^3\left[\ln 5 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot 5^n}(x^2)^n\right] \quad (3)$$

$$= \ln 5\cdot x^3 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\cdot 5^n}x^{2n}. \quad (1)$$