Exam 3

Name: ______________________________ Section: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Also, it helps you to show work on multiple choice problems so you can re-check your answers.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Find the average value of the function $f(x) = \cos^4 x \sin x$ on $0 \le x \le \pi$.

   A. $\frac{2}{5\pi}$

   B. $\frac{5}{2\pi}$

   C. 0

   D. 4

   E. 5

2. (5 points) The base of a solid S is the circle $x^2 + y^2 = 4$ and the parallel cross-sections perpendicular to the base are equilateral triangles. Remember that if an equilateral triangle has side length s, its area is $\frac{\sqrt{3}}{4}s^2$. Which of the following integrals correctly computes the volume of the solid?

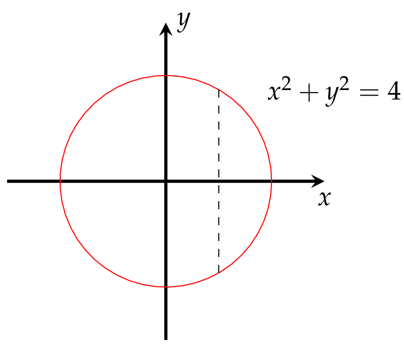
   A. $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$

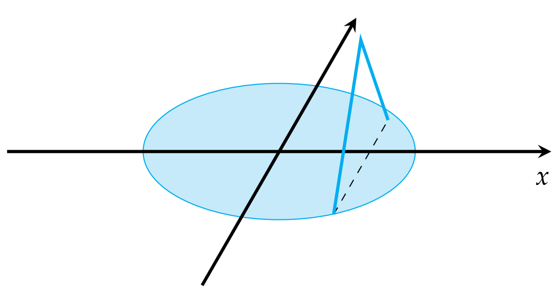
   B. $\int_{-2}^{2} \sqrt{3}(4 - x^2)\, dx$

   C. $\int_{-2}^{2} \frac{\sqrt{3}}{2}(4 - x^2)\, dx$

   D. $\int_{-2}^{2} (4 - x^2)^2\, dx$

   E. $\int_{-2}^{2} \sqrt{1 + 4x^2}\, dx$





3. (5 points) Which of the following is the Cartesian equation for the parametric curve $x(t) = \sin t, y(t) = 1 - \cos t$?

   A. $x^2 + y^2 = 1$

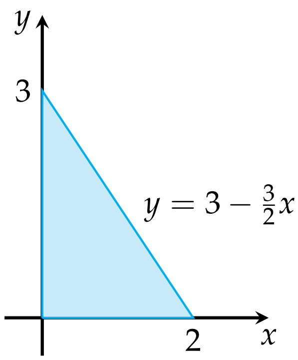
   B. $x^2 + (y+1)^2 = 1$

   C. $x^2 + (y-1)^2 = 1$

   D. $(x-1)^2 + y^2 = 1$

   E. $x^2 + y^2 = 2xy$

4. (5 points) What is the center of mass of the triangular region shown?



   A. $(1/3, 2/3)$

   B. $(3, 2)$

   C. $(2, 3)$

   D. $(2/3, 1)$

   E. $(1/3, 1/3)$

5. (5 points) Which of the following integrals computes the arc length of the curve $x = 1 + 3t^2$, $y = 4 + 2t^3$ for $0 \leq t \leq 1$?

A. $\int_0^1 \sqrt{1 + (6t)^2 + (36t^2)^2}\, dt$

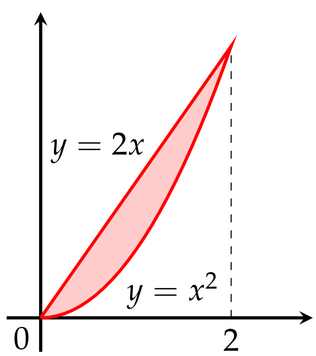
B. $\int_0^1 \sqrt{(1 + 3t^2)^2 + (4 + 2t^3)^2}\, dt$

C. $\int_0^1 \sqrt{3t^2 + 2t^3}\, dt$

D. $\int_0^1 \sqrt{2t^2 + 3t^3}\, dt$

E. $\int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$

6. (5 points) Which of the following integrals computes the volume of the solid obtained by rotating the region between the curves $y = x^2$ and $y = 2x$ about the x-axis?



A. $\int_0^2 2\pi(2x - x^2)\, dx$

B. $\int_0^2 2\pi(x^2 - 2x)\, dx$

C. $\int_0^2 \pi \left(4x^2 - x^4\right)\, dx$

D. $\int_0^2 \pi \left(x^4 - 4x^2\right)\, dx$

E. $\int_0^2 \pi \left(2x - x^2\right)^2\, dx$

7. (5 points) If $x = e^t \cos t$ and $y = e^t \sin t$, then $dy/dx =$

   A. $e^t \tan t$

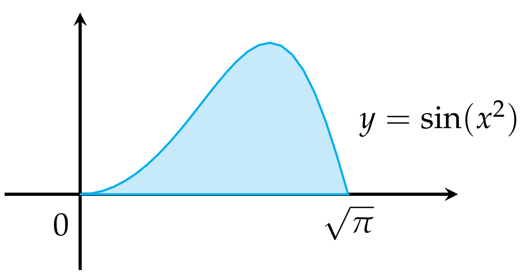
   B. $(\cos t + \sin t)/(\cos t - \sin t)$

   C. $(e^t \cos t)/(e^t \sin t)$

   D. $(\cos t - \sin t)/(\cos t + \sin t)$

   E. $(e^t \sin t)/(e^t \cos t)$

8. (5 points) Find the volume of the solid obtained by rotating the region shown in the figure about the y-axis.



   A. $\pi/2$

   B. π

   C. $3\pi/2$

   D. 2π

   E. 4π

9. (5 points) A surface S is obtained by rotating the curve

$$x = t\sin t,\ y = t\cos t, \quad 0 \le t \le \pi/2$$

about the x-axis. The area of the surface S is:

A. $S = \int_0^{\pi/2} 2\pi t\cos(t)\sqrt{1+t^2}\,dt$

B. $S = \int_0^{\pi} 2\pi t\cos(t)\sqrt{1+t^2}\,dt$

C. $S = \int_0^{\pi} 2\pi t\sin(t)\sqrt{1+t^2}\,dt$

D. $S = \int_0^{\pi} 2\pi t^2\sin(t)\,dt$

E. $S = \int_0^{\pi/2} 2\pi t^2\sin(t)\,dt$

10. (5 points) Find the center of mass of the three equal-mass particles shown in the figure at right.

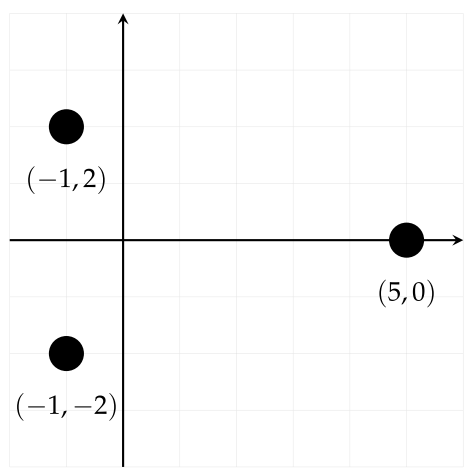
A. $(-1,0)$

B. $(0,0)$

C. $(1,0)$

D. $(2,0)$

E. $(3,0)$



Free Response Questions

11. (10 points) Find the points (x, y) on the curve $x = \cos t$, $y = \cos 3t$, $0 \le t \le \pi$, where the tangent line is horizontal.

Solution: First, note that

$$\frac{dx}{dt} = -\sin(t), \quad \frac{dy}{dt} = -3\sin(3t)$$

so

$$\frac{dy}{dx} = 3\frac{\sin(3t)}{\sin t}.$$

We want $\sin(3t) = 0$ so $3t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \ldots$. In $0 \le t \le \pi$ we have $t = 0,\ \pi/3,\ 2\pi/3,\ \pi$. At $t = 0,\ \pi$, the denominator is zero. So we take $t = \pi/3,\ 2\pi/3$.

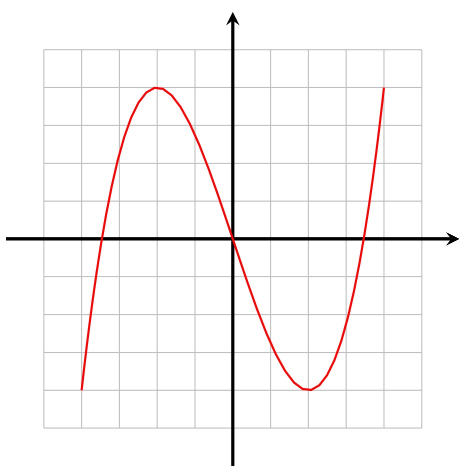
Now we compute

t	$x = \cos t$	$y = \cos(3t)$
$\pi/3$	$\frac{1}{2}$	-1
$2\pi/3$	$-\frac{1}{2}$	1

Hence, the points are $(x, y) = (1/2, -1),\ (-1/2, 1)$.

1 point each for correct expressions for dx/dt, dy/dt, and dy/dx; 1 point for stating that $\sin(3t) = 0$ at a horizontal tangent, and 1 point for stating (or implying) that $\sin t$ must be *nonzero*; 1 point for correctly deducing the values of t where horizontal tangents occur; 2 points each for the x and y-coordinates corresponding to these points (i.e., 2 points per correct (x, y) pair).

Here's a plot of the curve where you can see the horizontal tangents. Notice how the plot looks strangely like a cubic polynomial? There's a reason for this that you can figure out using trig identities.



12. (a) (5 points) Set up but do not evaluate an integral for the exact arc length of the curve

$$x = t\sin t,\ y = t\cos t, \quad 0 \le t \le 1$$

Solution: First

$$\frac{dx}{dt} = \sin t + t\cos t, \quad \frac{dy}{dt} = \cos t - t\sin t$$

so

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2} \\ &= \sqrt{1+t^2}\end{aligned}$$

Hence

$$s = \int_0^1 \sqrt{1+t^2}\,dt$$

1 point each for dx/dt and dy/dt; 2 points for correct computation of ds; 1 point for integral with correct limits.

(b) (5 points) Find dy/dx and d^2y/dx^2 for the curve $x = t^2+1$, $y = t^2+t$. For what values of t is the curve concave upward?

Solution: First,

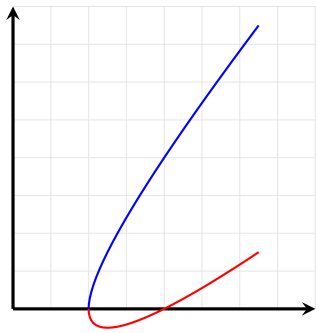
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}.$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{1}{(dx/dt)}\frac{d}{dt}\left(\frac{dy}{dx}\right) \\ &= \frac{1}{2t}\frac{d}{dt}\left(1+\frac{1}{2t}\right) = \frac{1}{2t}\left(-\frac{1}{2t^2}\right) = -\frac{1}{4t^3}\end{aligned}$$

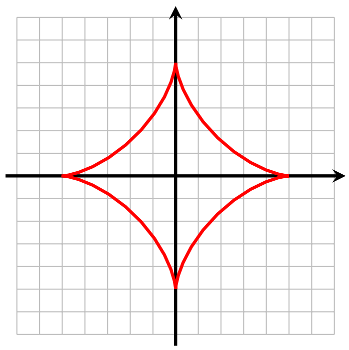
The curve is concave up when the second derivative is positive, i.e., for $t < 0$.

Here's a visual, where red is $t < 0$ and blue is $t > 0$.



1 point for dy/dx, 1 point for a correct formula for the second derivative (first line of displayed equation above), 2 points for correct computation of second derivative; 1 point for correct condition on t for the graph to be concave upward.

13. (a) (5 points) Find the area enclosed by the curve $x = a\cos^2\theta$, $y = a\sin^2\theta$.



Solution: We can use the fourfold symmetry, compute the area in one quadrant, and multiply by four at the end. Setting $y = a\sin^2\theta$, $dx = -2a\cos\theta\sin\theta$, and using $A = \int y\,dx$, we see that the area in the first quadrant is given by

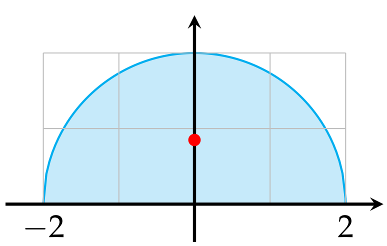
$$\begin{aligned} A &= \int_0^{\pi/2} 2a^2\sin^3\theta\cos\theta\,d\theta \\ &= 2a^2\int_0^1 u^3\,du \\ &= \frac{a^2}{2} \end{aligned}$$

(we took out the $-$ sign on dx because the area is positive; in this parameterization, $\theta = 0$ corresponds to $x = 1$, and $\theta = \pi/2$ corresponds to $x = 0$). Hence, the total area is $2a^2$.

Correct area formula $A = \int y\,dx$, 1 point; correct substitution for y and dx, 1 point each; correct set up of area integral including limits, 1 point; evaluation of integral, 2 points.

(b) (5 points) Find the center of mass of the region bounded by the circle $x^2 + y^2 = 4$ and the x-axis.

Solution:



By symmetry, the x-coordinate of the center of mass is $x = 0$. To find the y-coordinate of the center of mass, we compute (assuming density $\rho = 1$ and setting $f(x) = \sqrt{4 - x^2}$)

$$M = \int_{-2}^{2} f(x)\,dx = 2\pi$$

$$M_x = \int_{-2}^{2} \frac{1}{2}(4 - x^2)\,dx = \frac{16}{3}$$

so that $y = M_x/M = 8/3\pi \approx 0.8488$.
Hence $(x, y) = (0, 8/(3\pi))$.

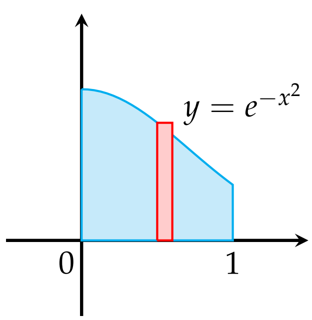
Symmetry argument, 1 point; correct computation of mass, 1 point; correct computation of moment, 2 points; answer, 1 point

14. Using the method of cylindrical shells, find the volume generated by rotating the region bounded by the curves

$$y = e^{-x^2}, \quad y = 0, \quad x = 0, \quad x = 1$$

about the y-axis.

Solution:



We'll integrate along the x axis from $x = 0$ to $x = 1$. A cylindrical shell at x has height e^{-x^2}, circumference $2\pi x$, and width dx, so

$$V = \int_0^1 2\pi x e^{-x^2}\, dx.$$

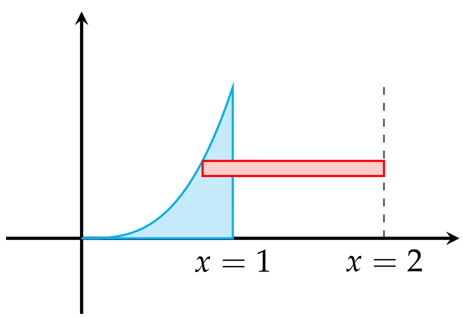
Using the substitution $u = x^2$, $du = 2x\, dx$, we compute

$$V = \int_0^1 \pi e^{-u}\, du$$
$$= \pi(1 - e^{-1})$$

Correct limits, 1 point; Identify radius, height, width correctly, 1 point each or 3 points total for correct integrand; set up volume integral correctly with limits, 1 point; computation of integral, 4 points; correct answer, 1 point

15. Let $\mathscr{R}$ be the region bounded by the curves $y = x^3$, $y = 0$, and $x = 1$. Find the volume obtained by rotating $\mathscr{R}$ about the line $x = 2$. You may use either the washer or shell method, but be sure to make clear which method you are using and illustrate with a careful sketch.

Solution:



We'll compute the volume by the washer method, integrating along the y-axis from $y = 0$ to $y = 1$. The inner radius of a washer is 1 and the outer radius is $1 + (1 - \sqrt[3]{y}) = 2 - \sqrt[3]{y}$. The region goes from $y = 0$ to $y = 1$. So, the integral for the volume is

$$V = \pi \int_0^1 \left[(2 - \sqrt[3]{y})^2 - 1 \right] \, dy.$$

Thus we get

$$\begin{aligned} V &= \pi \int_0^1 \left[\left(4 - 4y^{1/3} + y^{2/3} \right) - 1 \right] \, dy \\ &= \pi \left[4y - 3y^{4/3} + \frac{3}{5} y^{5/3} - y \right] \Big|_0^1 \\ &= \frac{3\pi}{5} \end{aligned}$$

I'll give a suggested breakdwon for the solution by the washer method; the grading term should determine an analogous breakdown for the shell method.

Identify region and limits of integration, 2 points; compute inner and outer radii correctly, 1 point each; correct volume integral including limits, 2 points; computation of integral, 3 points; answer, 1 point