Exam 3

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

1. (5 points) What is the average value of the function $|x-2|$ on the interval $[0,4]$?

   A. 4

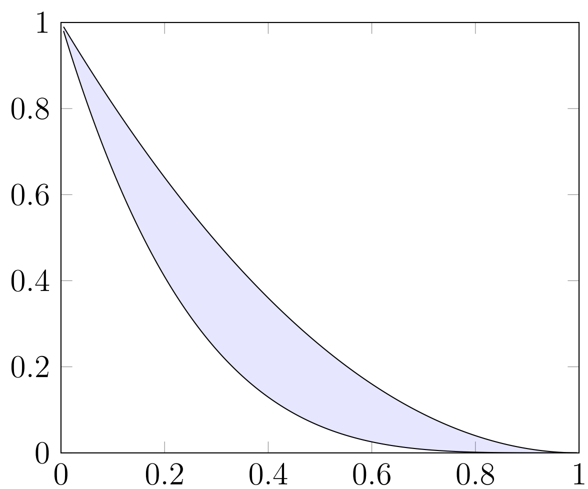
   B. $\frac{1}{2}$

   C. 1

   D. $\frac{1}{4}$

   E. 8

2. (5 points) The region bounded by $y=(x-1)^2$ and $y=(x-1)^4$ is shown below.



Consider the solid obtained by rotating this region around **the line y = 1**. Using the **disks/washers** method, which integral will compute the volume of this solid?

   A. $\int_0^1 \pi\left((1-(x-1)^4)^2-(1-(x-1)^2)^2\right)dx$

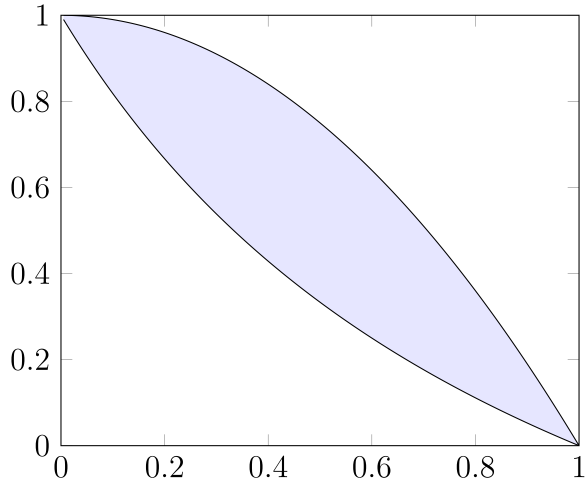
   B. $\int_0^1 2\pi x^2\sqrt{4x^2+16x^6}dx$

   C. $\int_0^1 \left((x-1)^4-(x-1)^2\right)dx$

   D. $\int_0^1 \pi\left((1-x^2)^2-(1-x^4)^2\right)dx$

   E. $\int_0^1 2\pi x\left((x-1)^2-(x-1)^4\right)dx$

3. (5 points) The region bounded by the curves $y = \dfrac{1-x}{1+x}$ and $y = 1 - x^2$ is shown below.



Consider the solid obtained by rotating this region about the **y-axis**. Using the **shell** method, which integral will compute the volume of this solid?

A. $\displaystyle\int_0^1 \pi\left((1-x^2)^2 - \frac{(1-x)^2}{(1+x)^2}\right) dx$

B. $\displaystyle\int_0^1 2\pi x\left(1 - x^2 - \frac{1-x}{1+x}\right) dx$

C. $\displaystyle\int_0^1 2\pi x(1-x^2)dx$

D. $\displaystyle\int_0^1 \pi x\left(\frac{(1-x)^2}{(1+x)^2} - (1-x^2)^2\right) dx$

E. $\displaystyle\int_0^1 \pi\left((1-x)^2 - (1+x)^2\right) dx$

4. (5 points) Which integral computes the **arc length** of the curve defined by the graph of the function $f(x) = \sqrt{1-4x^2}$ where $0 \le x \le 1$?

A. $\displaystyle\int_0^1 \sqrt{1 + x^2\sqrt{1-4x^2}}dx$

B. $\displaystyle\int_0^1 16x^2\sqrt{1+\sqrt{1-4x^2}}dx$

C. $\displaystyle\int_0^1 x\sqrt{4x^2+1}dx$

D. $\displaystyle\int_0^1 \sqrt{(1+4x^2)^2 + 16x^2}dx$

E. $\displaystyle\int_0^1 \sqrt{1 + \frac{16x^2}{1-4x^2}}dx$

5. (5 points) Which integral below computes the **surface area** of the surface obtained by revolving the graph of the function $f(x) = e^{2x}$ for $1 \leq x \leq 3$ around the **x-axis**?

A. $\int_1^3 2\pi e^{2x}\sqrt{1+4e^{4x}}dx$

B. $\int_1^3 \pi x\sqrt{e^{2x}+e^{4x}}dx$

C. $\int_1^3 \pi\left(1-e^{2x}\right)dx$

D. $\int_1^3 2\pi x\sqrt{e^{2x}-1}dx$

E. $\int_1^3 \sqrt{1+4e^{4x}}dx$

6. (5 points) Three masses are located in the plane: 7 kilograms at $(-7,1)$, 4 kilograms at $(11,-1)$, and 2 kilograms at $(1,-8)$. Find the center of mass of this system.

A. $(-3,-13)$

B. $(\frac{44}{13},-\frac{16}{13})$

C. $(\frac{1}{13},\frac{3}{13})$

D. $(-3,-\frac{11}{13})$

E. $(-\frac{3}{13},-1)$

7. (5 points) Which of the following is a parametrization of a curve defined by the equation:

$$\left(\frac{x-1}{2}\right)^2 - \left(\frac{y-3}{5}\right)^2 = 1$$

A. $x(t) = 2\cos(t) - 1, \quad y(t) = 5\sin(t) - 3$

B. $x(t) = 2\sec(t), \quad y(t) = 5\tan(t)$

C. $x(t) = \sec(2t) - 1, \quad y(t) = \tan(2t) - 3$

D. $x(t) = 2\sec(t) + 1, \quad y(t) = 5\tan(t) + 3$

E. $x(t) = \cos(5t) + 1, \quad y(t) = 2\sin(2t) + 3$

8. (5 points) Eliminate the parameter t to find a Cartesian equation satisfied by the curve parametrized by $x(t) = \sqrt{t^2+1}$, $y(t) = 1 - t$.

A. $x^2 + y^2 = 2$

B. $(x+1)^2 + y^2 = 1$

C. $x^2 - (y-1)^2 = 1$

D. $(x-1)^2 + (y-1)^2 = 2$

E. $x^2 - y^2 = 2$

9. (5 points) Find the slope of the tangent line to the curve parametrized by $x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t)$ at the point $(x, y) = (\frac{\pi}{2} - 1, 1)$.

    A. $\frac{\pi}{2} - 1$

    B. $\frac{2}{\pi}$

    C. $\frac{1}{2}$

    D. $\frac{\pi - 1}{2}$

    E. 1

10. (5 points) Which integral below computes the **surface area** of the surface obtained by revolving the curve parametrized by $x(\theta) = 2\cos(\theta)$, $y(\theta) = 3\sin(\theta)$ $0 \leq \theta \leq \pi$ around the **x-axis**?

    A. $\int_0^{2\pi} 3\pi \cos(\theta)\sqrt{4\sin^2(\theta) + 9\cos^2(\theta)}d\theta$

    B. $\int_0^{\pi} 6\pi \sin(\theta)\sqrt{4\sin^2(\theta) + 9\cos^2(\theta)}d\theta$

    C. $\int_0^{\pi} \sqrt{4\sin^2(\theta) + 9\cos^2(\theta)}d\theta$

    D. $\int_0^{\pi} 30\pi \sin(\theta)d\theta$

    E. $\int_0^{2} \pi 6\pi\theta \left(2\sin(\theta) + 3\cos(\theta)\right) d\theta$

Free Response Questions

11. Let C be the curve parametrized by the functions

$$x(t) = \frac{1-t^2}{1+t^2},$$

$$y(t) = \frac{2t}{1+t^2},$$

(a) (2 points) Find a number a so that $x(a) = 0, y(a) = 1$.

> **Solution:** The number a must satisfy $\frac{1-a^2}{1+a^2} = 0$, so $1 - a^2 = 0$, $1 = a^2$, and $a = \pm 1$. We also need $\frac{2a}{1+a^2} = 1$, so we take $a = 1$.

(b) (4 points) Find the **slope** of the tangent line to C at the point $(0, 1)$.

> **Solution:** We must compute $y'(t)$ and $x'(t)$. We have $x'(t) = \frac{(-2t)}{1+t^2} + \frac{(1-t^2)(2t)}{-(1+t^2)^2}$ and $y'(t) = \frac{2}{1+t^2} + \frac{(2t)^2}{-(1+t^2)^2}$. Evaluating at $a = 1$ gives $x'(1) = -1$ and $y'(1) = 0$, so the slope is $\frac{y'(1)}{x'(1)} = 0$

(c) (4 points) Set up <u>but do not evaluate</u> the integral to find the arc length of the piece of C given by $0 \le t \le 1$ (you may assume that the curve is traced only once).

> **Solution:** The arclength integral is $\int_a^b ((x'(t))^2 + (y'(t))^2)^{\frac{1}{2}} dt$ so we get:
>
> $$L = \int_0^1 ((\frac{(-2t)}{1+t^2} + \frac{(1-t^2)(2t)}{-(1+t^2)^2})^2 + (\frac{2}{1+t^2} + \frac{(2t)^2}{-(1+t^2)^2})^2)^{\frac{1}{2}} dt$$

12. Let S be the region in the plane bounded by $y = x^2$ and the line $y = 4$. Assume that S has uniform density $\rho = 1$.

(a) (8 points) Find the total mass M and the moments M_y and M_x for S.
Clearly label each of your answers.

Solution:

$$M = \rho \int_a^b f(x)dx = \int_{-2}^2 4 - x^2 dx =$$
$$[4x - \frac{1}{3}x^3]_{-2}^2 = 2(8 - \frac{8}{3}) = \frac{32}{3}$$

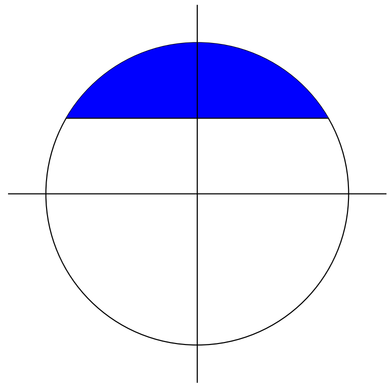
$$M_y = \rho \int_a^b xf(x)dx = \int_{-2}^2 x(4 - x^2)dx =$$
$$[2x^2 - \frac{1}{4}x^4]_{-2}^2 = 0$$

$$M_x = \frac{\rho}{2} \int_a^b f(x)^2 dx = \frac{1}{2} \int_{-2}^2 (4 - x^2)^2 dx =$$
$$\frac{1}{2}[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5]_{-2}^2 = (32 - \frac{64}{3} + \frac{32}{5}) = \frac{256}{15}$$

(b) (2 points) Find the center of mass of S.

Solution: The center of mass is $(\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{8}{5})$

13. Let R be the region in the plane given by those points with y coordinate larger than 1 that are contained in the circle of radius 2 centered at the origin. Let V be the solid obtained by revolving R around the **x-axis**.



(a) (4 points) Write an integral which computes the volume of V using the disk/washer method.

Solution: The bounds are the solutions to $1 = (4 - x^2)^{\frac{1}{2}}$, which are $\pm\sqrt{3}$.

$$\pi \int_{-\sqrt{3}}^{\sqrt{3}} (\sqrt{(}4 - x^2))^2 - (\sqrt{1})^2 dx = \pi \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 dx$$

(b) (4 points) Write an integral which computes the volume of V using the cylindrical shells method.

Solution: In this case our bounds run from $y = 1$ to $y = 2$. The height of the shell is $2(4 - y^2)^{\frac{1}{2}}$

$$2\pi \int_1^2 2y(4 - y^2)^{\frac{1}{2}} dy$$

(c) (2 points) Evaluate one of the integrals above to obtain the volume of V.

Solution: The first integral gives:

$$\pi \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 dx = \pi[3x - \frac{1}{3}x^3]_{-\sqrt{3}}^{\sqrt{3}} = 2\pi[3\sqrt{3} - \frac{1}{3}(\sqrt{3})^3]$$

14. Let L be the arc parametrized by $x(t) = \sqrt{t-1}$, $y(t) = 2t+3$, $0 \le t \le 1$.

(a) (5 points) Set up but do not evaluate an integral which computes the arc length of L.

Solution: The arclength is given by $L = \int_a^b ((x'(t))^2 + (y'(t))^2)^{\frac{1}{2}} dt$. We need to compute derivatives:

$$x'(t) = \frac{1}{2}(t-1)^{-\frac{1}{2}} = \frac{1}{2(t-1)^{\frac{1}{2}}}$$

$$y'(t) = 2$$

So we get

$$L = \int_0^1 (\frac{1}{2(t-1)^{\frac{1}{2}}})^2 + (2)^2)^{\frac{1}{2}} dt = \int_0^1 (\frac{1}{4t-4} + 4)^{\frac{1}{2}} dt$$

(b) (5 points) Set up but do not evaluate an integral which computes the area of the surface S obtained by revolving L around the **y-axis**.

Solution: The surface area is given by $S = \int_a^b 2\pi x(t)((x'(t))^2 + (y'(t))^2)^{\frac{1}{2}} dt$, so we get

$$\int_0^1 2\pi\sqrt{t-1}(\frac{1}{4t-4} + 4)^{\frac{1}{2}} dt = 2\pi \int_0^1 (\frac{1}{4} + 4t - 4)^{\frac{1}{2}} dt$$

15. Let K be a cone of height 10 and radius 5.

(a) (5 points) Find a function giving the area of the cross-section of K at height $0 \leq z \leq 10$.

Solution: We need to use similar triangles to find the radius of the slice of this cone at height z. We have $\frac{10-z}{r(z)} = \frac{10}{5}$, so $r(z) = \frac{1}{2}(10-z)$. The area of the cross-section is then $\pi r(z)^2 = \frac{\pi}{4}(10-z)^2$.

(b) (3 points) Set up an integral which computes the volume of K.

Solution: We integrate cross-sectional area: $\int_0^{10} \frac{\pi}{4}(10-z)^2 dz$

(c) (2 points) Find the volume of K.

Solution: This integral gives $\pi(5)^2(\frac{10}{3})$.