MA 114 — Calculus II Fall 2015
Exam 4 December 15, 2015

Name: ______________________________

Section: ______________________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer. Each problem has one correct answer and checking more than one answer will not receive credit.
- **Free Response Questions:** Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E

Exam Scores

Question	Score	Total
MC		36
10		17
11		15
12		18
13		16
14		18
Total		120

Unsupported answers for the free response questions may not receive credit!

There are some trigonometric identities given on the last page.

Record the correct answer to the following problems on the front page of this exam.

1. Consider the differential equation

$$y'(t) = 3y(t)\big(10 - y(t)\big).$$

Which of the following initial conditions guarantees that the solution $y(t)$ satisfies $\lim_{t\to\infty} y(t) = 10$?

logistic equation

$y'(t) = 30\,y(t)\left(1 - \frac{y(t)}{10}\right)$

$k = 30,\ A = 10$

A. $y(0) = 0$.

B. $y(0) = -1$.

C. $y(0) = -3$.

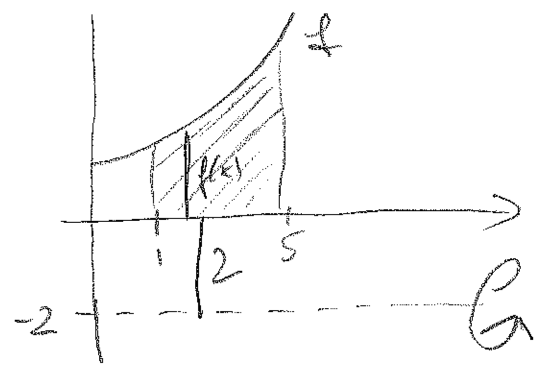
D. $y(0) = 1$.

E. All of the above.

2. Consider the graph of $f(x) = x^2 + 2$ and the region

enclosed by the x-axis, the graph of f, and the lines $x = 1$ and $x = 5$.

Which of the following integrals represents the volume of the solid obtained by revolving this region about the line $y = -2$?



A. $2\pi \int_1^5 (x^2+2)^2\,dx$.

B. $\pi \int_1^5 (x^2+4)^2\,dx$.

C. $\int_1^5 x^4\,dx$.

D. $\pi \int_1^5 \left((x^2+2)^2 - 4\right)dx$.

$\pi \int_1^5 \left((f(x)+2)^2 - 2^2\right)dx$

E. $\pi \int_1^5 \left((x^2+4)^2 - 4\right)dx$.

Record the correct answer to the following problems on the front page of this exam.

3. Substituting $x = 3\tan(\theta)$ in the integral $\int \sqrt{x^2+9}\,dx$ leads to which of the following integrals?

$\int 9\sec^3\theta\,d\theta$

$x = 3\tan\theta$
$dx = 3\sec^2\theta\,d\theta$
$\sqrt{x^2+9} = 3\sec\theta$

A. $\displaystyle\int 3\sec(\theta)\,d\theta.$

B. $\displaystyle\int 9\sec^3(\theta)\,d\theta.$

C. $\displaystyle\int 3\sec^3(\theta)\,d\theta.$

D. $\displaystyle\int 9\sec^2(\theta)\tan(\theta)\,d\theta.$

E. $\displaystyle\int 3\tan(\theta)\sec(\theta)\,d\theta.$

4. Which of the following differential equations is separable?

A. $y' - 9y^2 = x.$

B. $x^3y' - 9y^2 = 0.$ $\quad y' = \frac{1}{x^3}\cdot 9y^2$ separable

C. $yy' + x + y = 0.$

D. $(x+y)y' - \sqrt{y} = 0.$

E. $y' = \sin(xy).$

Record the correct answer to the following problems on the front page of this exam.

5. Let f be any differentiable function such that f' is continuous. Which of the following is true for the arc length of the graph of f over the interval $[-3, 2]$?

A. The arc length may be infinite.

B. The arc length may be negative.

C. The arc length is at most 5.

D. The arc length is at least 5.

E. None of the above.

$$\int_{-3}^{2} \underbrace{\sqrt{1+f'(x)^2}}_{\geq 1}\, dx$$

$$\geq \int_{-3}^{2} 1 \cdot dx = 5$$

6. Which of the following curves is described in polar coordinates by the equation

$$r\cos(\theta) = 2, \quad \text{where} \quad -\pi/2 < \theta < \pi/2.$$

A. The vertical line $x = 2$.

B. The horizontal line $y = 2$.

C. The circle of radius 2 centered at the origin.

D. The circle of radius 2 centered at the point $(0, 1)$.

E. The circle of radius 2 centered at the point $(1, 0)$.

$r = 2\sec\theta$

$x = r\cos\theta = 2$

$y = r\sin\theta = 2\tan\theta$

Record the correct answer to the following problems on the front page of this exam.

7. Which of the following statements is true for the differential equation

$$x^2y' = (x-1)(y-5)^2\left(4 - \frac{2y}{7}\right)?$$

A. $y = 0$ is a solution.

B. $y = 5$ and $y = 14$ are solutions.

C. $y = 5$ is the only constant solution.

D. The equation has no constant solutions.

E. None of the above.

8. Evaluate $\int 2xg(x)dx$.

$\int \underbrace{2x}_{v'}\underbrace{g(x)}_{u}\,dx$ $\quad u' = g'(x)$ $\quad v = x^2$

$x^2 g(x) - \int x^2 g'(x)\,dx$

A. $2xg(x) - \int 2xg'(x)dx$.

B. $x^2g(x) - \int x^2g'(x)dx$.

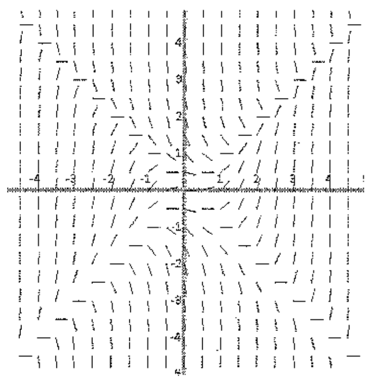
C. $xg(x) - x^2g'(x)$.

D. $2xg(x) - \int x^2g''(x)dx$.

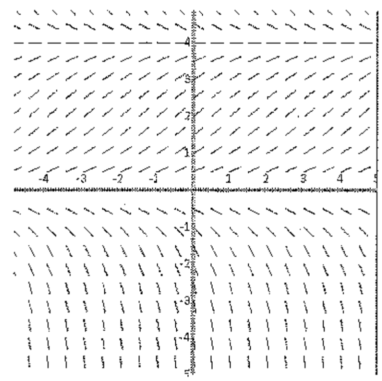
E. $x^2g'(x)$.

Record the correct answer to the following problems on the front page of this exam.

9. Which of the following is the slope field for the differential equation $y' = x^2 - 2y$?



A



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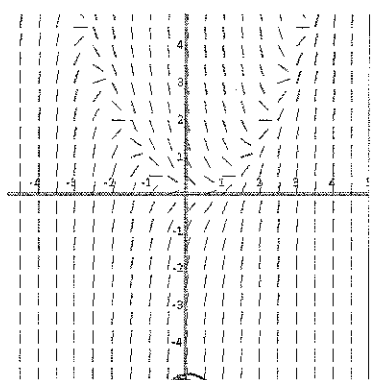

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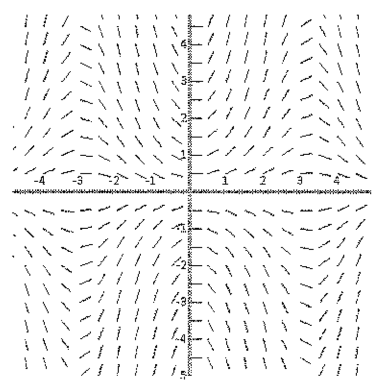
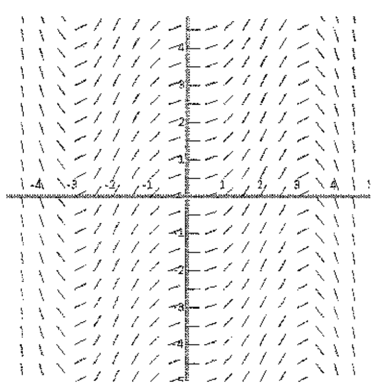


D



E

$y' = 0$
$\Rightarrow 2y = x^2$
$\Rightarrow y = \frac{1}{2}x^2$
Horizontal line segment along the graph of $y = \frac{1}{2}x^2$

Free Response Questions: Show your work!

10. Consider the differential equation $y' = y^2(e^x - 3x^2)$.

(a) Find the general solution.

The equation is separable.

$$\int \frac{1}{y^2}\,dy = \int (e^x - 3x^2)\,dx$$

$$-y^{-1} = e^x - x^3 + C$$

$$\boxed{y = \frac{1}{x^3 - e^x - C}}$$

(b) Find the solution satisfying the initial condition $y(0) = \frac{1}{2}$.

$$\frac{1}{2} = y(0) = \frac{1}{-1-C} \Rightarrow C = -3$$

$$\boxed{y = \frac{1}{x^3 - e^x + 3}}$$

11. Consider the power series

$$F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}.$$

(a) Determine the radius of convergence.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1} \cdot n \cdot 3^n}{(n+1)3^{n+1} \cdot x^n}\right| = \frac{1}{3}\frac{n}{n+1}|x|$$

$$\xrightarrow[n\to\infty]{} \frac{1}{3}|x| \overset{!}{<} 1. \text{ Hence } |x|<3.$$

Radius of convergence is $\boxed{3}$

(b) Determine the interval of convergence.

The series converges on the interval $(-3,3)$ plus possibly at the endpoints.

$\underline{x=3}$: $\sum_{n=1}^{\infty} \frac{3^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(harmonic series)

$\underline{x=-3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

(alternating harmonic series).

Interval of convergence: $\boxed{[-3,3)}$

Free Response Questions: Show your work!

12. Consider the parametrized curve

$$x(t) = t^2 - 9, \; y(t) = t^2 - 8t, \text{ where } -\infty < t < \infty.$$

(a) Show that the point $(16, -15)$ is on the curve and find all value(s) of t that correspond to this point.

$t^2 - 9 = 16 \Rightarrow t^2 = 25 \Rightarrow t = \pm 5$

$y(5) = 25 - 40 = -15$

$y(-5) = 25 + 40 = 65$

$\boxed{\text{the point } (16,-15) \text{ corresponds to the parameter value } t=5.}$

(b) Find the slope of the tangent line to the curve at the point $(16, -15)$.

$$\frac{y'(t)}{x'(t)} = \frac{2t-8}{2t}$$

$$\frac{y'(5)}{x'(5)} = \frac{2}{10} = \frac{1}{5}. \quad \boxed{\text{slope} = \frac{1}{5}}$$

(c) Find all points on the curve where the curve has a vertical tangent line.

the slope is infinite if $2t = 0$ and $2t - 8 \neq 0$.
this is the case for $t = 0$.
Hence the curve has a vertical tangent line at the point $(x(0), y(0)) = \underline{(-9, 0)}$.

(d) Show that there is no point on the curve where the tangent line has slope 1.

$$\frac{2t-8}{2t} = 1 \Rightarrow 2t - 8 = 2t \Rightarrow -8 = 0 \; ↯$$

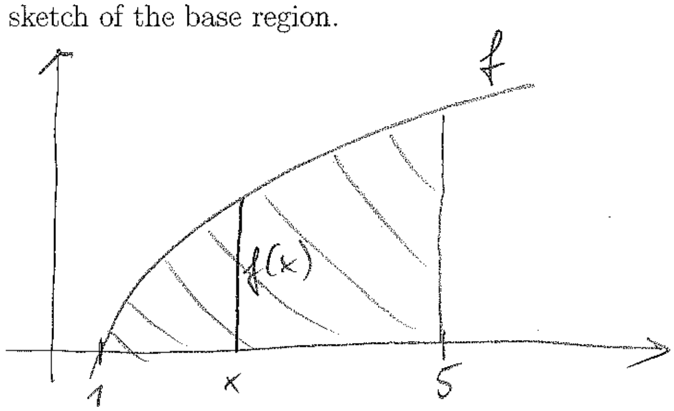
thus there is no t for which $\frac{2t-8}{2t} = 1$.

13. A solid is given whose base is the region

enclosed by the x-axis, the graph of $f(x) = 2\ln(x)$ and the line $x = 5$,

and whose cross sections perpendicular to the x-axis are rectangles of height $\frac{1}{x}$.

(a) Give a sketch of the base region.



(b) Compute the volume of the solid.

$$V = \int_1^5 f(x) \cdot \frac{1}{x}\,dx = \int_1^5 2\ln(x)\frac{1}{x}\,dx$$

$$\underset{\substack{u = \ln(x) \\ du = \frac{1}{x}dx}}{=} 2\int_0^{\ln(5)} u\,du = u^2 \Big|_0^{\ln(5)}$$

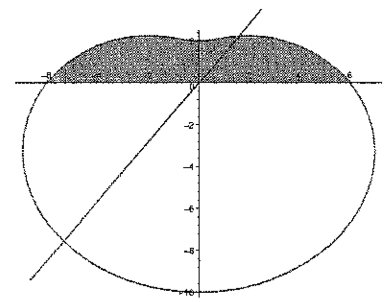
$$= \boxed{\ln(5)^2}$$

Free Response Questions: Show your work!

14. The picture shows the curve described in polar coordinates by the equation

$$r = 6 - 4\sin(\theta),\ 0 \le \theta \le 2\pi$$

$r > 0$ always.
Hence shaded region
corresponds to
$0 \le \theta \le \pi$



(a) Compute the area of the shaded region.

$$\frac{1}{2}\int_0^{\pi} (6-4\sin\theta)^2 \, d\theta = \int_0^{\pi} \Big(18 - 24\sin\theta + \underbrace{8\sin^2\theta}_{4(1-\cos(2\theta))}\Big) d\theta$$

$$= \int_0^{\pi} (22 - 24\sin\theta - 4\cos(2\theta))\, d\theta$$

$$= (22\theta + 24\cos\theta - 2\sin(2\theta))\Big|_0^{\pi} = 22\pi - 24 - 24$$

$$= \boxed{22\pi - 48}$$

(b) Find the polar coordinates of both intersection points of the curve with the line given by the equation $\theta = \frac{\pi}{4}$.

To obtain all intersection points (r, θ), where $r > 0$, we need to consider $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

$\theta = \frac{\pi}{4}$: $r = 6 - 4\sin\left(\frac{\pi}{4}\right) = 6 - 2\sqrt{2}$

$\theta = \frac{5\pi}{4}$: $r = 6 - 4\sin\left(\frac{5\pi}{4}\right) = 6 + 2\sqrt{2}$

the points are $\boxed{\left(6-2\sqrt{2}, \frac{\pi}{4}\right), \left(6+2\sqrt{2}, \frac{5\pi}{4}\right)}$.

Some Trigonometric Identities

$\sin^2(x) + \cos^2(x) = 1$
$\tan^2(x) + 1 = \sec^2(x)$
$\sin^2(x) = \frac{1}{2}\big(1 - \cos(2x)\big)$
$\cos^2(x) = \frac{1}{2}\big(1 + \cos(2x)\big)$
$\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\frac{d}{dx}\tan(x) = \sec^2(x)$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$