MA 114 — Calculus II Spring 2014
Exam IV May 7, 2014

Name: ______________________________

Section: ______________________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:** Show all your work on the page of the problem. Show all your work. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	B	~~C~~	D	E
2	A	B	C	~~D~~	E
3	A	B	~~C~~	D	E
4	A	~~B~~	C	D	E
5	A	B	C	~~D~~	E

Exam Scores

Question	Score	Total
MC		20
6		12
7		12
8		16
9		12
10		12
11		16
Total		100

Unsupported answers for the free response questions may not receive credit!

Record the correct answer to the following problems on the front page of this exam.

1. The polar coordinates (r, θ) of the point P with rectangular coordinates $(x, y) = (-\sqrt{2}, \sqrt{2})$ can be expressed as:

A. $(2, \frac{\pi}{4})$

B. $(-2, \frac{\pi}{4})$

C. $(2, \frac{3\pi}{4})$

D. $(-2, \frac{3\pi}{4})$

E. $(2, \frac{7\pi}{4})$

2. The motion of a particle in the x-y plane is given parametrically by the equations

$$x(t) = 3t - \cos(3t), \quad y(t) = 3t - \sin(3t), \quad 0 \leq t \leq \pi,$$

were the units of distance and time are meters and seconds, respectively. The speed (in m/s) of the particle at $t = \frac{\pi}{3}$ is:

A. 3

B. $\sqrt{5}$

C. 0

D. $3\sqrt{5}$

E. $5\sqrt{3}$

Record the correct answer to the following problems on the front page of this exam.

3. Which of the following integrals calculates the area of the region enclosed by the curve $r = 2\cos\theta - 1$ for $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$?

A. $\frac{1}{2}\int_0^{2\pi} (2\cos\theta - 1)^2\, d\theta$

B. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos\theta - 1)\, d\theta$

C. $\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos\theta - 1)^2\, d\theta$

D. $2\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos\theta - 1)\sqrt{(2\cos\theta - 1)^2 + (2\sin\theta)^2}\, d\theta$

E. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{(2\cos\theta - 1)^2 + (2\sin\theta)^2}\, d\theta$

4. Find the most general form of the anti-derivative of $x\cos(2x)$ by integration by parts. Below, C denotes an arbitrary constant.

A. $x\sin(2x) + C$

B. $\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$

C. $\frac{1}{2}\cos(2x) - \frac{1}{2}\sin(2x) + C$

D. $x\cos(2x) + C$

E. None of the above

Record the correct answer to the following problems on the front page of this exam.

5. Which of the following answers is true for the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$?

A. It is divergent.

B. It is absolutely convergent, but divergent.

C. It is convergent, but not absolutely convergent.

D. It is absolutely convergent and convergent.

E. None of the above.

Free Response Questions: Show your work!

6. (12 points) Find the area of surface obtained by rotating the parametric curve $c(t) = (\sin^2(t), \cos^2(t))$ around the x-axis, for $0 \le t \le \frac{\pi}{2}$.

$x'(t) = 2 \sin t \cos t$ — 2

$y'(t) = -2 \cos t \sin t$ — 2

$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{4 \sin^2 t \cos^2 t + 4 \cos^2 t \sin^2 t}$

$= 2\sqrt{2} \sin t \cos t, \quad 0 \le t \le \pi/2$ — 2

$\text{Area} = 2\pi \int_0^{\frac{\pi}{2}} y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$ — 2

$= 2\pi \int_0^{\frac{\pi}{2}} 2\sqrt{2} \sin t \cos t \cos^2 t\, dt$

$= 4\sqrt{2}\pi \int_0^{\frac{\pi}{2}} \cos^3 t \sin t\, dt$

$\overset{u = \cos t}{\underset{du = -\sin t\, dt}{=\!=}} 4\sqrt{2}\pi \int_0^1 u^3\, du$

$= 4\sqrt{2}\pi \left. \frac{u^4}{4} \right|_0^1$

— 4

$= \pi\sqrt{2}$

Free Response Questions: Show your work!

7. (12 points) Use separation of variables to find the solution of the initial value problem:

$$\frac{dx}{dt} = (1+t^2)(x^2+1), \quad x(0) = 0.$$

$\left(\textbf{Hint: } \frac{d}{dx}\arctan x = \frac{1}{x^2+1}\right).$

• $\frac{dx}{x^2+1} = (1+t^2)\,dt$ — 3

• $\int \frac{dx}{x^2+1} = \int (1+t^2)\,dt$ — 1

$\Rightarrow \tan^{-1}(x) = t + \frac{t^3}{3} + C$ — 5

Since $x(0)=0$, we have

$\tan^{-1}(0) = 0 + \frac{0^3}{3} + c$ — 2

and hence $c = 0$

$\Rightarrow \tan^{-1}(x) = t + \frac{1}{3}t^3$

or $\boxed{x = \tan\left(t + \frac{1}{3}t^3\right)}$ — 1

Free Response Questions: Show your work!

8. (16 points)

(a) (8 points) Determine whether or not the series $\sum_{n=0}^{\infty} \frac{n^2}{e^n}$ converges. Make sure to state the test(s) that you use, and verify that their assumptions are satisfied.

We use the Ratio test: — 1

Since $\rho = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{1}{e} = \frac{1}{e} < 1,$ 5

we conclude by the Ratio test that $\sum_{n=0}^{\infty} \frac{n^2}{e^n}$ is convergent. 2

(b) (8 points) Find the Maclaurin series (the Taylor series centered at $c = 0$) of the function $f(x) = \frac{1}{1+3x}$ and find its radius of convergence (**Hint**: You can use the geometric series $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$).

Since $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ holds for $-1<t<1$

substituting t by $-3x$, we obtain

$$\boxed{\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-3)^n x^n}$$ — 6

The radius of convergence is determined by $|3x| < 1$ or $\boxed{|x| < \frac{1}{3}}$ — 2

Free Response Questions: Show your work!

9. (12 points) Find the arc length of the parametrized curve $(x(t), y(t)) = (3t^2+1, 4t^3-10)$ for $1 \le t \le 4$ (**Give the exact answer**).

$$\begin{cases} x'(t) = 6t & -2 \\ y'(t) = 12t^2 & -2 \end{cases}$$

$$\text{Arc length} = \int_1^4 \sqrt{x'(t)^2 + y'(t)^2}\, dt \quad -2$$

$$= \int_1^4 \sqrt{(6t)^2 + (12t^2)^2}\, dt$$

$$= \int_1^4 6t\sqrt{1+4t^2}\, dt \quad -2$$

$$\xrightarrow[du = 8t\,dt]{u = 1+4t^2} \frac{3}{4}\int_5^{65} \sqrt{u}\, du$$

$$= \frac{3}{4} \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_5^{65}$$

$$= 8\left(65^{3/2} - 5^{3/2}\right) \quad -4$$

4

Free Response Questions: Show your work!

10. (12 points) Find the total area enclosed by the cardioid $r = 1 - \cos\theta$ for $0 \le \theta \le 2\pi$ (**Give the exact answer**) (**Hint**: recall the half-angle formula $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$).

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\, d\theta \quad - 4$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta \quad - 1$$

$$= \underbrace{\frac{1}{2}\left(\theta - 2\sin\theta \Big|_0^{2\pi}\right)}_{-2} + \underbrace{\frac{1}{2}\int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\, d\theta}_{-2}$$

$$= \frac{1}{2}\left(2\pi - 2(\sin 2\pi - \sin 0)\right) + \frac{1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} \quad - 2$$

$$= \pi + \frac{1}{4}\left(2\pi + \frac{1}{2}(\sin(4\pi) - \sin 0)\right)$$

$$= \frac{3\pi}{2} \quad - 1$$

Free Response Questions: Show your work!

11. (16 points)

(a) (8 points) Use the method of partial fractions to evaluate the integral: $\displaystyle\int \frac{x}{x^2 - 5x + 6}\,dx$.

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \quad - (4)$$

$$A(x-3) + B(x-2) = x \Rightarrow \begin{cases} A+B=1 \\ 3A+2B=0 \end{cases}$$

$$\Rightarrow A = -2,\ B = 3 \quad - (2)$$

$$\int \frac{x}{x^2-5x+6} = \int \left(\frac{-2}{x-2} + \frac{3}{x-3} \right) dx \quad - (2)$$

$$= -2\ln|x-2| + 3\ln|x-3| + C$$

(b) (8 points) Use trigonometric substitution to evaluate the integral $\displaystyle\int \sqrt{1-x^2}\,dx$ (**Hint**: set $x = \sin\theta$ and apply the half-angle formula from problem 10).

$$x = \sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad - (2)$$

$$dx = \cos\theta\, d\theta \quad - (1)$$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta \quad - (1)$$

$$\int \sqrt{1-x^2}\,dx = \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \quad - (3)$$

$$(\theta = \sin^{-1} x) \quad = \frac{1}{2}\sin^{-1} x + \frac{1}{4}\sin(2\sin^{-1} x) + C \quad - (1)$$