Exam 4

Name: ______________________ Section and/or TA: ______________

Last Four Digits of Student ID: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	15	(BONUS) 16	Total Score
50	10	10	10	10	10	10	110

Multiple Choice Questions

1. The solution to the initial value problem

$$\frac{dy}{dx} = xe^y, \quad y(0) = 0$$

is:

A. $y = \ln\left(1 - x^2\right)$

B. $y = -\ln\left(1 - \frac{x^2}{2}\right)$

C. $y = \ln\left(1 - \frac{x^2}{2}\right)$

D. $y = 1 - e^{-x^2/2}$

E. $y = 1 - \frac{e^{-2x}}{2}$

2. The equation

$$9x^2 - 18x + 4y^2 = 27$$

is the equation of

A. An ellipse with $a = 3$, $b = 2$, **and foci at** $(1, \pm\sqrt{5})$

B. An ellipse with $a = 3$, $b = 2$, and foci at $(\pm\sqrt{5}, 1)$

C. An ellipse with $a = 3$, $b = 2$, and foci at $(-1, \pm\sqrt{5})$

D. An ellipse with $a = 3$, $b = 2$, and foci at $(0, \pm\sqrt{5})$

E. An ellipse with $a = 2$, $b = 2$, and foci at $(0, 0)$

3. What is the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$?

   A. 0 (the series converges for no nonzero x)

   B. ∞ (the series converges for all x)

   C. 1

   D. $\sqrt{2}$

   E. 3

4. Which of the following curves has the polar description $r = 5\cos\theta$?

   A. A circle of radius 5 with center at $(5, 0)$

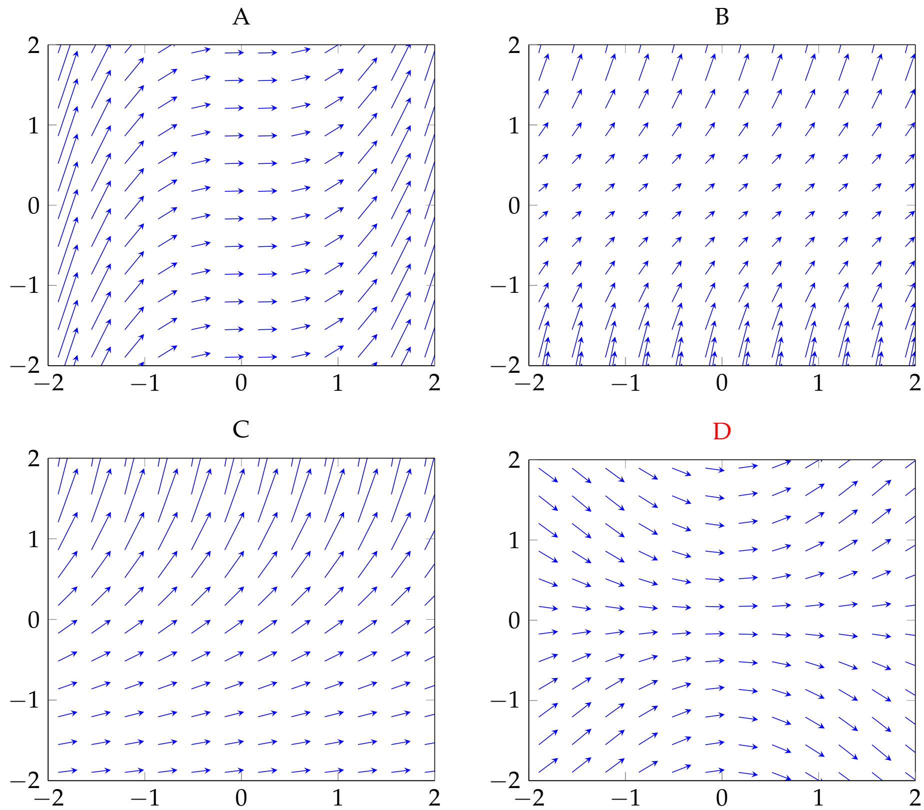
   B. A circle of radius $5/2$ with center at $(0, 5/2)$

   C. A circle of radius $5/2$ with center at $(5/2, 0)$

   D. A circle of radius $5/2$ with center at $(-5/2, 0)$

   E. A circle of radius $5/2$ with center at $(0, -5/2)$

5. Which of the following is the direction field for the differential equation $y' = \sin x \sin y$?



6. A population grows according to the logistic equation

$$\frac{dP}{dt} = 0.04P\left(1 - \frac{P}{1200}\right), \quad P(0) = 60.$$

Which of the following is true about this model (there is only one correct choice)?

A. The carrying capacity is 60 and the growth constant k is 0.04.

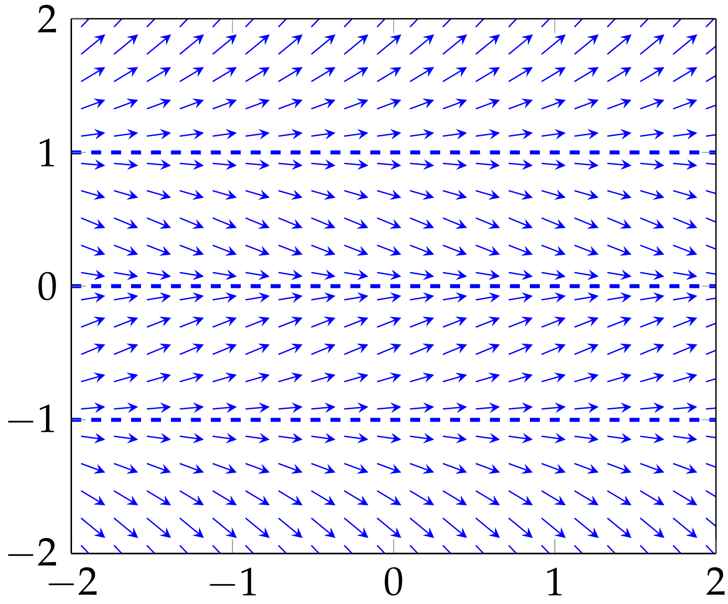
B. The carrying capacity is 1200 and the growth constant k is 60

C. The carrying capacity is 1200 and the growth constant k is 0.04

D. The initial population is 60 and the carrying capacity is 25

E. As $t \to \infty$ the population goes to zero.

7. Below is the direction field for a differential equation of the form $y' = f(y)$. Which of the following statements is correct (there is only one correct choice)?



A. $y = 0$, $y = 1$, and $y = -1$ are all stable equilibria

B. $y = 0$ and $y = 1$ are stable equilibria, while $y = -1$ is an unstable equilibrium

C. $y = 1$ and $y = -1$ are unstable equilibria, but $y = 0$ is a stable equilibrium

D. $y = 1$ and $y = -1$ are the only equilibria, and both are stable

E. $y = 0$ is the only equilibrium, and is unstable

8. What is the slope of the tangent line to the graph of $r = 2\cos\theta$ at $\theta = \pi/3$?

   A. 1

   B. -1

   C. $\frac{2\sqrt{3}}{2}$

   D. $-1/\sqrt{3}$

   E. $1/\sqrt{3}$

9. Which of the following is the correct form for the partial fraction decomposition of $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2}$?

   A. $\dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+2}$

   B. $\dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+2} + \dfrac{(Ex+F)^2}{(x^2+2)^2}$

   C. $\dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+2} + \dfrac{Ex+F}{(x^2+2)^2}$

   D. $\dfrac{Ax^2+Bx+C}{x^2-x+1} + \dfrac{Cx+D}{x^2+2} + \dfrac{Ex+F}{(x^2+2)^2}$

   E. None of the above

10. A function has MacLaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^n = 1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \cdots$$

Find $f'''(0)$.

   A. $1/7$

   B. $24/7$

   C. $-1/7$

   D. $6/7$

   E. $-6/7$

Free Response Questions

11. The equation

$$9y^2 - 4x^2 - 36y - 8x = 4$$

defines a hyperbola.

(a) (4 points) By completing the square in x and y, put this equation in standard form.

Solution:

$$9(y^2 - 4y) - 4(x^2 + 2x) = 4$$
$$9(y - 2 - 4y + 4) - 4(x^2 + 2x + 1) = 4 + 36 - 4$$
$$9(y-2)^2 - 4(x+1)^2 = 36$$
$$\frac{(y-2)^2}{4} - \frac{(x+1)^2}{9} = 1$$

(b) (3 points) Find the foci, vertices, and asymptotes of the curve.

Solution: This curve is a hyperbola shifted by $(-1, 2)$. The standard form hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

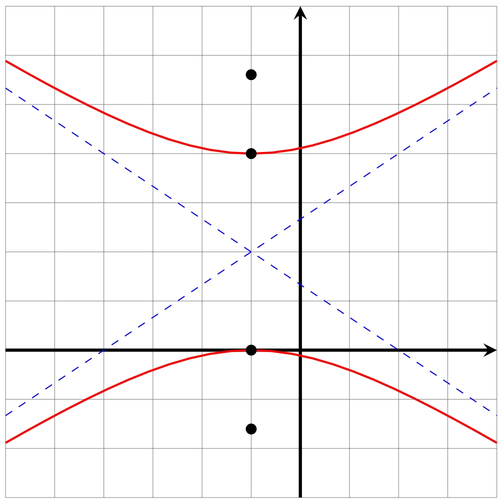
has foci $(\pm c, 0)$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(a/b)x$. In our case, $a = 2$, $b = 3$, and hence $c^2 = 3^2 + 2^2 = 13$ or $c = \sqrt{13}$. Incorporating the shift we get:

Foci: $(-1, 2 \pm \sqrt{13})$

Vertices: $(-1, 0)$ and $(-1, 4)$

Asymptotes: $y - 2 = \pm(2/3)(x + 1)$

(c) (3 points) Sketch the curve on the axes provided. Label the foci and the vertices, and sketch the asymptotes.



Solution: Vertices are plotted at $(-1, 0)$ and $(-1, 4)$, and foci are shown at $(-1, 2 - \sqrt{13})$ and $(-1, 2 + \sqrt{13})$, The asymptotes are plotted as blue dashed lines.

12. Newton's law of cooling states that the rate of heat loss of a body is proportional to the difference between its temperature and the temperature of its surroundings. Thus if $T(t)$ is the temperature of the object and T_0 is the temperature of its surroundings,

$$\frac{dT}{dt} = -k\left(T(t) - T_0\right)$$

where k is a rate constant. A cup of coffee at 95° C is placed in a 20° C room and reaches a temperature of 57.5° after $1/2$ hour.

(a) (2 points) Set up the initial value problem (differential equation and initial condition) for the temperature of the coffee as a function of time in hours.

Solution:

$$\frac{dT}{dt} = -k\left(T(t) - 20\right), \quad T(0) = 95$$

(b) (4 points) Using the method of separation of variables, solve this equation for $T(t)$ up to a constant of integration and the constant k.

Solution:

$$\begin{aligned} \frac{dT}{T-20} &= -kdt \\ \ln(T-20) &= -kt + C \\ T - 20 &= Ce^{-t} \\ T &= 20 + Ce^{-kt} \end{aligned}$$

(c) (4 points) Find the rate constant k and the constant of integration, and state the solution formula.

Solution: At time 0, the coffee is at 95° C, so

$$\begin{aligned} 95 &= 20 + C \\ C &= 75 \end{aligned}$$

After 1/2 hour, the coffee's temperature is 75° C so

$$\begin{aligned} 57.5 &= 20 + 75e^{-(1/2)k} \\ 37.5 &= 75e^{-(1/2)k} \\ \frac{1}{2} &= e^{-(1/2)k} \\ -\ln 2 &= -\frac{1}{2}k \\ k &= 2\ln 2 \end{aligned}$$

The solution formula is then

$$T(t) = 20 + 75e^{-(2\ln 2)t}$$

(d) (2 points) How long does it take for the coffee to cool to 38.75° C?

Solution: We solve the equation

$$\begin{aligned} 38.75 &= 20 + 75e^{-(2\ln 2)t} \\ 18.75 &= 75e^{-(2\ln 2)t} \\ \frac{1}{4} &= e^{-(2\ln 2)t} \\ -\ln 4 &= -(2\ln 2)t \\ t &= 1 \end{aligned}$$

13. Find the radius of convergence and the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n.$$

Solution: Using the ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n5^n}{(n+1)5^{n+1}} x\right|$$
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x}{5}\right|$$

so the radius of convergence of the series is 5.

To find the interval of convergence, we check the endpoints $x = -5$ and $x = 5$.

(a) $x = -5$: The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}(-5)^n = -\sum_{n=1}^{\infty} \frac{1}{n}$$

is a harmonic series which diverges.

(b) $x = 5$: The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}(5)^n = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is an alternating series which converges. Hence, the interval of convergence is $(-5, 5]$.

14. The goal of this problem is to find the area that lies inside the curves $r = 3\sin\theta$ and $r = 3\cos\theta$.

(a) (4 points) Find the equation of each of these curves in Cartesian coordinates, and identify the curves.

Solution: The first curve has Cartesian equation

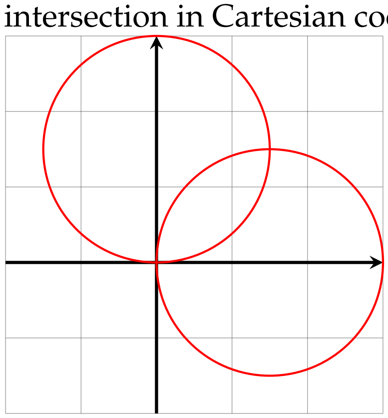
$$\begin{aligned} x^2 + y^2 &= 3y \\ x^2 + y^2 - 3y &= 0 \\ x^2 + \left(y - \frac{3}{2}\right)^2 &= \frac{9}{4} \end{aligned}$$

while the second curve has Cartesian equation

$$\begin{aligned} x^2 + y^2 &= 3x \\ x^2 - 3x + y^2 &= 0 \\ \left(x - \frac{3}{2}\right)^2 + y^2 &= \frac{9}{4} \end{aligned}$$

Hence, the first curve is a circle of radius $3/2$ with center $(0, 3/2)$, and the second curve is a circle of radius $3/2$ with center $(3/2, 0)$.

(b) (2 points) Graph the two curves on the axes provided, and find their points of intersection in Cartesian coordinates.



Solution: From the graph, one of the points of intersection is $(0, 0)$. The other point of intersection has $\sin\theta = \cos\theta = \pi/4$ so $r = 3/\sqrt{2}$ and $\theta = \pi/4$. Hence $(x, y) = \left(\frac{3}{2}, \frac{3}{2}\right)$.

(c) (4 points) Set up, but do not evaluate, an integral for the area that lies inside of both curves.

Solution: The area that lies inside both curves is bounded for $0 \leq \theta \leq \pi/4$ by the curve $r = 3\sin\theta$ and for $\pi/4 \leq \theta \leq \pi/2$ by the curve $r = 3\cos\theta$. So, the total area is given by

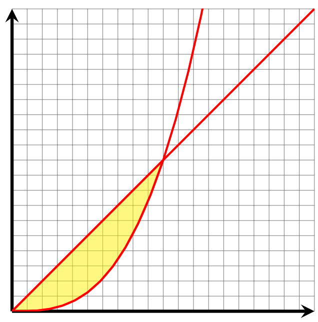
$$A = \int_0^{\pi/4} \frac{1}{2} 9\sin^2\theta \, d\theta + \int_{\pi/4}^{\frac{\pi}{2}} 9\cos^2\theta \, d\theta$$

By symmetry each of these integrals is equal so we also have

$$A = \int_0^{\pi/4} 9 \sin^2 \theta \, d\theta.$$

15. Suppose $\mathscr{R}$ is the region bounded by the curves $y = x^3$, $y = x$ with $x \geq 0$.

(a) Sketch the curves and the region $\mathscr{R}$ on the axes provided, and use the given equations to solve for the points of intersection.



Solution: The points of intersection are $(0, 0)$ and $(1, 1)$.

(b) Set up an integral for the volume of the region obtained by rotating $\mathscr{R}$ about the x-axis. Be sure to state whether you are using the disc method, the shell method, or the washer method.

Solution:

$$V = \int_0^1 \pi \left(x^2 - x^6\right)\, dx$$

(c) Compute the integral.

Solution:

$$V = \pi \left[\frac{x^3}{3} - \frac{x^7}{7}\right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{7}\right) = \frac{4\pi}{21}.$$

16. (OPTIONAL BONUS QUESTION) (10 points) This problem concerns using Taylor series to find limits. Solutions using L'Hospital's rule will receive no credit!

(a) Use the Taylor series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

to find

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x}.$$

Solution:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ e^{-x} &= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots \\ e^x - e^{-x} &= 2x + \frac{1}{3}x^3 + \dots \\ \frac{e^x - e^{-x}}{2x} &= 1 + \frac{1}{6}x^2 + \dots \end{aligned}$$

Hence, $\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} = 1$.

(b) Use the Taylor series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

to find

$$\lim_{x \to 0} \frac{\cos(2x) - 1}{x^2}.$$

Solution:

$$\begin{aligned} \cos(2x) &= 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \dots \\ \frac{\cos(2x) - 1}{x^2} &= \frac{1}{x^2}\left(-\frac{4x^2}{2} + \frac{16x^4}{24} - \dots\right) \\ &= -2 + \frac{2}{3}x^2 + \dots \end{aligned}$$

so

$$\lim_{x \to 0} \frac{\cos(2x) - 1}{x^2} = -2.$$