MA 114 Worksheet #01: Integration by parts

1. For each of the following integrals, determine if it is best evaluated by integration by parts or by substitution. If the integral should be evaluated by substitution, give the substition you would use. You do not need to find the anti-derivatives.

   (a) $\int x\cos(x^2)\,dx$,

   (b) $\int e^x \sin(x)\,dx$,

   (c) $\int \frac{\ln(\arctan(x))}{1+x^2}dx$,

   (d) $\int xe^{x^2}\,dx$

2. Find the following indefinite integrals using integration by parts:

   (a) $\int x^2\sin(x)\,dx$,

   (b) $\int (2x+1)e^x\,dx$,

   (c) $\int 2x\arctan(x)\,dx$,

   (d) $\int \ln(x)\,dx$

   (e) $\int x^5\ln(x)\,dx$

   (f) $\int e^x\cos x\,dx$

   (g) $\int x\ln(1+x)\,dx$ Hint: Make a substitution first, then try integration by parts.

3. Evaluate the definite integral $\int_0^3 x\sin(3-x)\;dx$.

4. Let $f(x)$ be a twice differentiable function with $f(1)=2$, $f(4)=7$, $f'(1)=5$ and $f'(4)=3$. Evaluate $\int_1^4 xf''(x)\,dx$

5. If $f(0)=g(0)=0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx.$$

MA 114 Worksheet #02: Special Trigonometric Integrals

1. Compute the following integrals:

   (a) $\int \sin x \sec^2 x dx$

   (b) $\int \sin^3 x dx$

   (c) $\int_0^{\pi/2} \cos^2(x)\, dx$

   (d) $\int \sqrt{\cos x} \sin^3 x\, dx$

   (e) $\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right)\, d\theta$

   (f) $\int_0^{\pi/2} (2 - \sin\theta)^2\, d\theta$

   (g) $\int 4 \sin^2 x \cos^2 x\, dx$

   (h) $\int \cos^5 x\, dx$.

2. Find the anti-derivative $\int \cot(x)\, dx$. Hint: Substitute $u = \sin(x)$.

3. Evaluate $\int \sin x \cos x\, dx$ by four methods:

   (a) the substitution $u = \cos x$;

   (b) the substitution $u = \sin x$;

   (c) the identity $\sin 2x = 2 \sin x \cos x$;

   (d) integration by parts

   Explain the different appearances of the answers.

4. Find the area of the region bounded by the curves $y = \sin^2 x$ and $y = \sin^3 x$ for $0 \leq x \leq \pi$.

MA 114 Worksheet #03: Trigonometric Substitution

1. Use the trigonometric substitution $x = \sin(u)$ to find $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$.

   Remark: This exercise verifies one of the basic anti-derivatives we learned in Calculus I. On an exam, you would be expected to know this anti-derivative and would not be expected to show work to evaluate the anti-derivative by substitution.

2. Compute the following integrals:

   (a) $\displaystyle\int_0^2 \frac{u^3}{\sqrt{16-u^2}}du$

   (b) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}dx$

   (c) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

   (d) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

   (e) $\displaystyle\int \frac{1}{(1+x)^2}\,dx$

   (f) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}dx.$

3. Evaluate the following integrals. One may be easily evaluated by substitution $u = 1+x^2$ and for the other use an appropriate trigonometric subsitution.

$$\int \frac{\sqrt{1+x^2}}{x}\,dx \qquad \int \frac{x}{\sqrt{1+x^2}}\,dx$$

4. (a) Evaluate the integral $\displaystyle\int_0^r \sqrt{r^2-x^2}\,dx$ using trigonometric substitution.

   (b) Use your answer to part a) to verify the formaula for the area of a circle of radius r.

5. Let $r > 0$. Consider the identity

$$\int_0^s \sqrt{r^2-x^2}\,dx = \frac{1}{2}r^2 \arcsin(s/r) + \frac{1}{2}s\sqrt{r^2-s^2}$$

   where $0 \leq s \leq r$.

   (a) Plot the curves $y = \sqrt{r^2-x^2}$, $x = s$, and $y = \dfrac{x}{s}\sqrt{r^2-x^2}$.

   (b) Using part (a), verify the identity geometrically.

   (c) Verify the identity using trigonometric substitution.

MA 114 Worksheet #04: Integration by partial fractions

1. Write out the general form for the partial fraction decomposition but do not determine the numerical value of the coefficients.

   (a) $\dfrac{1}{x^2+3x+2}$

   (b) $\dfrac{x+1}{x^2+4x+4}$

   (c) $\dfrac{x}{(x^2+1)(x+1)(x+2)}$

   (d) $\dfrac{2x+5}{(x^2+1)^3(2x+1)}$

2. Based on your work in the previous question, can you conjecture (guess) a relation between the degree of the denominator of the rational function and the number of coefficients in the partial fraction decomposition?

3. Find the partial fraction decomposition for the following rational functions.

   (a) $\dfrac{1}{x^2-4}$

   (b) $\dfrac{x^3}{x^2-4}$ Hint: This is not a proper rational function, so you will begin by dividing.

   (c) $\dfrac{x}{(x^2+1)(x+1)}$

4. Compute the following integrals.

   (a) $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

   (b) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

   (c) $\displaystyle\int \frac{x^3-2x^2+1}{x^3-2x^2}\,dx$

   (d) $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$

   (e) $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$

5. Compute

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$$

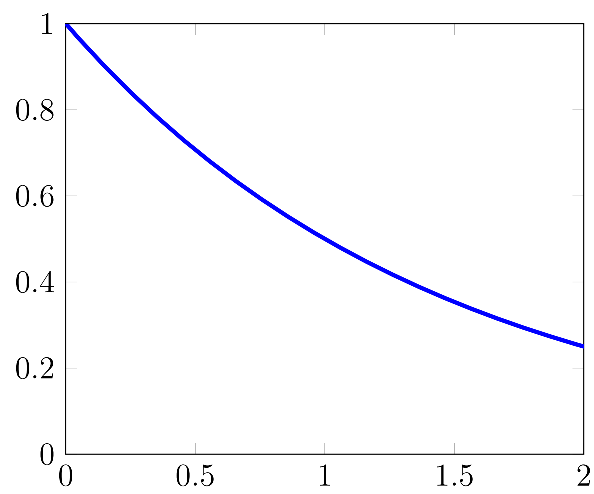
   by first making the substitution $u=\sqrt[6]{x}$.

MA 114 Worksheet #05: Numerical Integration

1. (a) Write down the Midpoint rule and illustrate how it works with a sketch.
   (b) Write down the Trapezoid rule, illustrate how it works with a sketch, and write down the error bound associated with it.
   (c) Use the error estimate for the trapezoid rule to find n so that you can approximate
   $$\int_0^1 \sin(2x)\,dx$$
   with an error less than 10^{-7}?

2. Use the Midpoint rule to approximate the value of $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx$ with $n = 4$. Draw a sketch to determine if the approximation is an overestimate or an underestimate of the integral.

3. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\,dx$, where f is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of sub- intervals were used in each case.
   (a) Which rule produced which estimate?
   (b) Between which two approximations does the true value of $\int_0^2 f(x)\,dx$ lie?



4. Draw the graph of $f(x) = \sin\left(\frac{1}{2}x^2\right)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\,dx$.
   (a) Use the graph to decide whether L_2, R_2, M_2, and T_2 underestimate or overestimate I.
   (b) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.
   (c) Compute L_5, R_5, M_5, and T_5. From the graph, which do you think gives the best estimate of I?

5. The velocity in meters per second for a particle traveling along the axis is given in the table below. Use the Midpoint rule and Trapezoid rule to approximate the total distance the particle traveled from $t = 0$ to $t = 6$.

t	$v(t)$
0	0.75
1	1.34
2	1.5
3	1.9
4	2.5
5	3.2
6	3.0

MA 114 Worksheet #06: Simpson's Rule and Improper Integrals

1. (a) Write down Simpson's rule and illustrate how it works with a sketch.
   (b) How large should n be in the Simpson's rule so that you can approximate

$$\int_0^1 \sin x \, dx$$

   with an error less than 10^{-7}?

2. Approximate the integral

$$\int_1^2 \frac{1}{x} \, dx$$

   using Simpson's rule. Choose n so that your error is certain to be less than 10^{-3}. Compute the exact value of the integral and compare to your approximation.

3. Simpson's Rule turns out to exactly integrate polynomials of degree three or less. Show that Simpson's rule gives the *exact* value of $\int_0^h p(x) \, dx$ where $h > 0$ and $p(x) = ax^3 + bx^2 + cx + d$. [Hint: First compute the exact value of the integral by direct integration. Then apply Simpson's rule with $n = 2$ and observe that the approximation and the exact value are the same.]

4. For each of the following, determine if the integral is proper or improper. If it is improper, explain why. Do ***not*** evaluate any of the integrals.

   (a) $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6} \, dx$
   (b) $\displaystyle\int_1^2 \frac{1}{2x - 1} \, dx$
   (c) $\displaystyle\int_1^2 \ln(x - 1) \, dx$
   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2} \, dx$
   (e) $\displaystyle\int_0^{\pi/2} \sec x \, dx$

5. For the integrals below, determine if the integral is convergent or divergent. Evaluate the convergent integrals.

   (a) $\displaystyle\int_{-\infty}^0 \frac{1}{2x - 1} \, dx$
   (b) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2} \, dx$
   (c) $\displaystyle\int_0^2 \frac{x - 3}{2x - 3} \, dx$
   (d) $\displaystyle\int_0^{\infty} \sin\theta \, d\theta$

6. Consider the improper integral

$$\int_1^{\infty} \frac{1}{x^p} \, dx.$$

Integrate using the generic parameter p to prove the integral converges for $p > 1$ and diverges for $p \leq 1$. You will have to distinguish between the cases when $p = 1$ and $p \neq 1$ when you integrate.

7. Use the Comparison Theorem to determine whether the following integrals are convergent or divergent.
   (a) $\displaystyle\int_1^\infty \frac{2 + e^{-x}}{x}\,dx$
   (b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^6 + x}}\,dx$

8. Explain why the following computation is wrong and determine the correct answer. (Try sketching or graphing the integrand to see where the problem lies.)

$$\begin{aligned}\int_2^{10} \frac{1}{2x-8}dx &= \frac{1}{2}\int_{-4}^{12} \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|x|\Big|_{-4}^{12} \\ &= \frac{1}{2}(\ln 12 - \ln 4)\end{aligned}$$

where we used the substitution

$$\begin{cases} u(x) = 2x - 8 \\ u(2) = -4 \qquad u(10) = 12 \\ \dfrac{du}{dx} = 2 \end{cases}$$

9. A manufacturer of light bulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the companys bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.
   (a) Make a rough sketch of what you think the graph of F might look like.
   (b) What is the meaning of the derivative $r(t) = F'(t)$?
   (c) What is the value of $\int_0^\infty r(t)\,dt$? Why?

MA 114 Worksheet #07: Sequences

1. (a) Give the precise definition of a **sequence**.
   (b) What does it mean to say that $\lim_{x \to a} f(x) = L$ when $a = \infty$? Does this differ from $\lim_{n \to \infty} f(n) = L$? Why or why not?
   (c) What does it means for a sequence to converge? Explain your idea, not just the definition in the book.
   (d) Sequences can diverge in different ways. Describe two distinct ways that a sequence can diverge.
   (e) Give two examples of sequences which converge to 0 and two examples of sequences which converges to a given number $L \neq 0$.

2. Write the first four terms of the sequences with the following general terms:
   (a) $\dfrac{n!}{2^n}$
   (b) $\dfrac{n}{n+1}$
   (c) $(-1)^{n+1}$
   (d) $\{a_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{3}{n}$.
   (e) $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2^{-n} + 2$.
   (f) $\{b_k\}_{k=1}^{\infty}$ where $b_k = \dfrac{(-1)^k)}{k^2}$.

3. Find a formula for the nth term of each sequence.
   (a) $\left\{\dfrac{1}{1}, -\dfrac{1}{8}, \dfrac{1}{27}, -\dfrac{1}{64}, \ldots\right\}$
   (b) $\left\{1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \ldots\right\}$
   (c) $\{1, 0, 1, 0, 1, 0, \ldots\}$
   (d) $\left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots, \right\}$

4. Suppose that a sequence $\{a_n\}$ is bounded above and below. Does it converge? If not, find a counterexample.

5. The limit laws for sequences are the same as the limit laws for functions. Suppose you have sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} a_n = 15$, $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 1$. Use the limit laws of sequences to answer the following questions.
   (a) Does the sequence $\left\{\dfrac{a_n \cdot c_n}{b_n + 1}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?
   (b) Does the sequence $\left\{\dfrac{a_n + 3 \cdot c_n}{2 \cdot b_n + 2}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?

MA 114 Worksheet #08: Review for Exam 01

1. Find the following antiderivatives

   (a) $\displaystyle\int x^2 \sin 2x\, dx$

   (b) $\displaystyle\int xe^{2x}\, dx$

   (c) $\displaystyle\int \frac{dx}{x^2+2x+10}$

   (d) $\displaystyle\int \frac{x+3}{(x-6)(x-3)}\, dx$

   (e) $\displaystyle\int \frac{3x+6}{x^2-10x+24}\, dx$

   (f) $\displaystyle\int \frac{3x^2+9x+8}{x^2(x+2)^2}\, dx$

   (g) $\displaystyle\int \sin^5 x \cos x\, dx$

   (h) $\displaystyle\int \sin^2 x\, dx$

   (i) $\displaystyle\int \frac{dx}{x\sqrt{x^2+9}}$

   (j) $\displaystyle\int \sqrt{16+4x^2}\, dx$

   (k) $\displaystyle\int x^3\sqrt{9-x^2}\, dx$

   (l) $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

   (m) $\displaystyle\int_1^\infty xe^{-2x}\, dx$

2. (a) Let $f(x) = e^{-x^2}$ and find $f''(x)$. Find the maximum value of $f''(x)$ on the interval $[0,1]$.

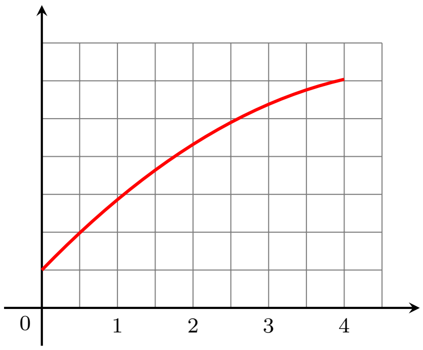
   (b) What is the smallest N we should use in the trapezoid rule to compute

   $$\int_0^1 e^{-x^2}\, dx$$

   accurate to within 0.0001?

3. Calculate M_6 and T_6 to approximate $\displaystyle\int_{-2}^1 e^{x^2}\, dx$.

4. Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is shown below. For any value of n, list the numbers L_n, R_n, M_n, and T_n in increasing order.



5. An airplane's velocity is recorded at 5-minute intervals during a 1 hour period with the following results, in miles per hour:

$$\begin{array}{ccccccc} 550, & 575, & 600, & 580, & 610, & 640, & 625, \\ 595, & 590, & 620, & 640, & 640, & 630 & \end{array}$$

Use Simpson's Rule to estimate the distance traveled during the hour.

6. For which values of p does the improper integral

$$\int_0^\infty \frac{dx}{(1+x)^p}\,dx$$

converge?