Chapter 1

Recursive Sequences

We have described a sequence in at least two different ways:

- a list of real numbers where there is a first number, a second number, and so on. We are interested in infinite sequences, so our lists do not end. Examples are $\{1, 2, 3, 4, 5, 6, \ldots\}$ or $\{2, 4, 8, 8, 8, 8, 8, 8, 16, \ldots\}$. The sequences we saw in the last section we were usually able to describe by some formula. This is not always the case.

- a *function* $a\colon \mathbb{N} \to \mathbb{R}$ where we denoted the output by $a(n) = a_n$. One example would be $a_n = n$. Others are $a_n = 2^n$, $a_n = 1/n$. Any function that is defined on the set of whole numbers gives us a *sequence*.

There is yet another way to describe a sequence. This process is known as *recursion*. Recursion is the process of choosing a starting term and repeatedly applying the same process to each term to arrive at the following term. Recursion requires that you know the value of the term or terms immediately before the term you are trying to find.

A recursive formula always has two parts:

1. the starting value for the first term a_0;

2. the recursion equation for a_n as a function of a_{n-1} (the term before it.)

Example 1.1. Consider the sequence given by $a_n = 2a_{n-1} + 1$ with $a_0 = 4$. The recursion function (or *recursion equation*) tells us how to find a_1, a_2, and so on.

$$\begin{aligned} a_1 &= 2a_1 + 1 = 2(4) + 1 = 9 \\ a_2 &= 2a_1 + 1 = 2(9) + 1 = 19 \\ a_3 &= 2a_2 + 1 = 2(19) + 1 = 39 \end{aligned}$$

What is a_{10}? Here the problem is that we have to find a_9 in order to find a_{10}, but to find a_9 we need a_8, but to find a_8 we need a_7, and so on.

Example 1.2. [Fibonacci sequence] Consider the following recursion equation.

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 1,\ F_1 = 1.$$

$$F_2 = F_1 + F_0 = 2$$
$$F_3 = F_2 + F_1 = 3$$

In fact, it is easier to list these out in a list by just adding the previous two terms to get the next term.

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots\}$$

The Fibonacci sequence has a long history in mathematics and you can find out more about it online at any number of websites. The Fibonacci sequence is named after the 13th-century Italian mathematician known as Fibonacci, who used it to solve a problem concerning the breeding of rabbits. This sequence also occurs in numerous applications in plant biology.

Example 1.3.

1. Write recursive equations for the sequence $\{5, 7, 9, 11, \ldots\}$.
2. Write recursive equations for the sequence $\{2, 4, 8, 16, \ldots\}$.
3. Write recursive equations for the sequence $\{1, 2, 6, 24, 120, 720, \ldots\}$.
4. Write recursive equations for the sequence $\{2, 3, 6, 18, 108, 1944, 209952, \ldots\}$

Exercises

1. What is the 5th term of the recursive sequence defined as follows: $a_1 = 5$, $a_n = 3a_{n-1}$?
2. What is the 5th term of the recursive sequence defined as follows: $a_1 = 2$, $a_n = 2a_{n-1} - 1$?
3. What is the 1st term of a recursive sequence in which $a_n = 4a_{n-1}$, if $a_4 = 192$?
4. Write recursive equations for the sequence $\{5, 11, 17, 23, \ldots\}$.
5. Write recursive equations for the sequence $\{3, 6, 12, 24, \ldots\}$.
6. Write recursive equations for the sequence $\{2, 4, 16, 256, 65536, \ldots\}$.
7. Write recursive equations for the sequence $\{2, 6, 14, 30, 62, \ldots\}$.
8. Write recursive equations for the sequence $\{3, 4, 7, 11, 18, 29, \ldots\}$.
9. Write recursive equations for the sequence $\{6561, 81, 9, 3, \ldots\}$.
10. Write the first five terms of the sequence in which $a_1 = 1$ and $a_n = 2a_{n-1} - 2$.
11. Write the first five terms of the sequence in which $a_1 = 2$ and $a_n = 5a_{n-1} - 5$.

Example 1.4. [Depreciation] Consider a situation in which the value of a car depreciates 10% per year. If the car is originally valued at $36,000, the following year it is worth 90% of $36,000, or $32,400. After another year, the value is 90% of $32,400, or $29,160. If we write the decreasing values as a list: 36,000, 32,400, 29,160…we have written a sequence - a sequence where each term depends on the value of the preceding term - a recursive sequence: $a_n = 0.9a_{n-1}$ with $a_0 = 36000$.

Two simple examples of recursive definitions are for arithmetic sequences and geometric sequences. An arithmetic sequence has a common difference, or a constant difference between each term.

$$a_n = a_{n-1} + d \quad \text{or} \quad a_n - a_{n-1} = d.$$

The common difference, d, is analogous to the slope of a line. In this case it is possible to find a formula for the nth term directly. This simplifies finding say the 42nd term.

A geometric sequence has a common ratio.

$$a_n = r \cdot a_{n-1} \quad \text{or} \quad \frac{a_n}{a_{n-1}} = r.$$

Again, in this case it is relatively easy to find a formula for the nth term: $a_n = a_0 r^n$. Thus, there are sequences that can be defined recursively, analytically, and those that can be defined in both manners.

Recursive sequences are sometimes called a *difference* equations. The recursive sequence in Example 1 is called a first-order difference equation because a_n depends on just the preceding term a_{n-1}, whereas the Fibonacci sequence is a second-order difference equation because F_n depends on the two preceding terms F_{n-1} and F_{n-2}. The general first-order difference equation is of the form $a_{n+1} = f(a_n)$ where f is some function. Why is it called a difference equation? The word difference comes from the fact that such equations are often formulated in terms of the difference between one term and the next: $\Delta a_n = a_{n+1} - a_n$. The equation $\Delta a_n = g(a_n)$ can be written as follows:

$$\begin{aligned} a_{n+1} - a_n &= g(a_n) \\ a_{n+1} &= a_n + g(a_n) = f(a_n) \end{aligned}$$

where $f(x) = x + g(x)$.

1.1 Limits of Recursive Sequences

In our previous discussion, we learned how to find $\lim_{n\to\infty} a_n$ when a_n is given explicitly as a function of n. How do you find such a limit when a_n is defined recursively.

When we define a first-order sequence $\{a_n\}$ recursively, we express a_{n+1} in terms of a_n and specify a value for a_1. We can then compute successive values of a_n, which might allow us to guess the limit if it exists. In some cases (as in the next example), we can find a solution of the recursion and then determine the limit (if it exists).

Example 1.5. Compute a_n for $n = 1, 2, \ldots, 6$ when $a_{n+1} = \frac{1}{4}a_n + \frac{3}{4}$ with $a_1 = 2$. Find a solution of the recursion, and then take a guess at the limiting behavior of the sequence.

$$\begin{aligned}
a_1 &= 2 \\
a_2 &= \frac{1}{4}a_1 + \frac{3}{4} = \frac{5}{4} = 1.25 \\
a_3 &= \frac{1}{4}a_2 + \frac{3}{4} = \frac{17}{16} = 1.0625 \\
a_4 &= \frac{1}{4}a_3 + \frac{3}{4} = \frac{65}{64} = 1.015625 \\
a_5 &= \frac{1}{4}a_4 + \frac{3}{4} = \frac{257}{256} = 1.00390625 \\
a_6 &= \frac{1}{4}a_5 + \frac{3}{4} = \frac{1025}{1024} = 1.0009765625
\end{aligned}$$

There seems to be a pattern, namely, that the denominators are powers of 4 and the numerators are just 1 larger than the denominators. We will try $a_n = \dfrac{4^{n-1}+1}{4^{n-1}}$ and check whether this is indeed a solution of the recursion.

First, we need to check the initial condition: $a_1 = (4^0 + 1)/4^0 = 2/1 = 2$. This agrees with the given initial condition. Next, we need to check whether a_n satisfies the recursion. Accordingly, we write

$$\begin{aligned}
a_{n+1} &= \frac{4^n + 1}{4^n} = 1 + \frac{1}{4} \cdot \left(\frac{1}{4^{n-1}}\right) = 1 + \frac{1}{4}\frac{1}{4^{n-1}} \\
a_n &= \frac{4^{n-1}+1}{4^{n-1}} \Rightarrow a_n = 1 + \frac{1}{4^{n-1}} \Rightarrow \frac{1}{4^{n-1}} = a_n - 1 \\
a_{n+1} &= 1 + \frac{1}{4}\frac{1}{4^{n-1}} = 1 + \frac{1}{4}(a_n - 1) = \frac{1}{4}a_n + \frac{3}{4}
\end{aligned}$$

which is the given recursion. We can now use our formula to find the limit. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^{n-1}+1}{4^{n-1}} = \lim_{n\to\infty}\left(1 + \frac{1}{4^{n-1}}\right) = 1.$$

since $\lim_{n\to\infty} \dfrac{1}{4^{n-1}} = 0$.

Finding an explicit expression for a_n as in the above example is often not possible, because solving recursions can be very difficult or even impossible. How, then, can we say anything about the limiting behavior of a recursively defined sequence?

The following procedure will allow us to identify *candidates* for limits: A **fixed point** of a function is a point x so that $f(x) = x$. For recursive sequences this translates as if the sequence $\{a_n\}$ is can be given as $a_{n+1} = f(a_n)$ and if a is a fixed point for $f(x)$, then if $a_n = a$ is equal to the fixed point for some k, then all successive values of a_n are also equal to a for $k > n$.

Now,if $a_{n+1} = g(a_n)$,then if $a_1 = a$ and a is a fixed point, it follows that $a_2 = g(a_1) = g(a) = a$, $a_3 = g(a_2) = g(a) = a$, and so on. That is, a fixed point satisfies the equation

$$a = g(a).$$

We will use this representation to find fixed points.

In the previous example, we had the recursion $a_{n+1} = \frac{1}{4}a_n + \frac{3}{4}$. Fixed points for the recursion thus would satisfy $a = \frac{1}{4}a + \frac{3}{4}$. Solve this equation for a.

$$\begin{aligned} a &= \frac{1}{4}a + \frac{3}{4} \\ \frac{3}{4}a &= \frac{3}{4} \\ a &= 1 \end{aligned}$$

We find that $a = 1$. In the above example this fixed point is also the limiting value of the sequence. This will not always be the case: A fixed point is only a *candidate* for a limit; a sequence does not have to converge to a given fixed point (unless a_0 is already equal to the fixed point). The next two examples illustrate convergence and non-convergence, respectively.

Example 1.6. Assume that $\lim_{n\to\infty} a_n$ exists for

$$a_{n+1} = \sqrt{3a_n} \text{ with } a_0 = 2.$$

Find $\lim_{n\to\infty} a_n$.

Since the problem tells us that the limit exists, we don't have to worry about existence. We may assume that $\lim_{n\to\infty} a_n = A$. The problem that remains is to identify the limit. To do this we need to note that if $\lim_{n\to\infty} a_n = A$ then it is true that $\lim_{n\to\infty} a_{n+1} = A$, since these are exactly the same sequence. Now, we compute the fixed points. We solve

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{3a_n} \\ A &= \sqrt{3A} \end{aligned}$$

This has two solutions, namely, $A = 0$ and $A = 3$. When $a_0 = 2$, we have $a_n > 2$ for all $n = 1, 2, 3, \ldots$, so we can exclude $A = 0$ as the limiting value. This leaves only one possibility, and we conclude that

$$\lim_{n\to\infty} a_n = 3.$$

Consider some successive values of a_n, which we collect in the following table (accurate to two decimals):

n	0	1	2	3	4	5	6	7
a_n	2	2.45	2.71	2.85	2.92	2.96	2.98	2.99

These values suggest that the limit is indeed 3.

Example 1.7. Let $a_{n+1} = \dfrac{3}{a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of a_n when a_0 is not equal to a fixed point.

To find the fixed points, we need to solve

$$a = \frac{3}{a}.$$

This equation is equivalent to $a^2 = 3$; hence, $a = \pm\sqrt{3}$. These are the two fixed points. If $a_0 = \sqrt{3}$, then $a_1 = \sqrt{3}$, $a_2 = \sqrt{3}$,and so on, and likewise, if $a_0 = -\sqrt{3}$,then $a_1 = -\sqrt{3}$, $a_2 = -\sqrt{3}$, and so on.

Let's start with a value that is not equal to one of the fixed points — say, $a_0 = 2$. Using the recursion, we find that

$$\begin{aligned} a_1 &= \frac{3}{a_0} = \frac{3}{2} \\ a_2 &= \frac{3}{a_1} = \frac{3}{\frac{3}{2}} = 2 \\ a_3 &= \frac{3}{a_2} = \frac{3}{2} \\ a_4 &= \frac{3}{a_3} = \frac{3}{\frac{3}{2}} = 2 \end{aligned}$$

and so on — successive terms alternate between 2 and 3/2.

Try another initial value, say, $a_0 = 3$. Then

$$\begin{aligned} a_1 &= \frac{3}{a_0} = \frac{3}{3} = 1 \\ a_2 &= \frac{3}{a_1} = \frac{3}{1} = 3 \\ a_3 &= \frac{3}{a_2} = \frac{3}{3} = 1 \\ a_4 &= \frac{3}{a_3} = \frac{3}{1} = 3 \end{aligned}$$

and so on. Successive terms now alternate between 3 and 1. In this specific example alternating between two values, one of which is the initial value, happens with any initial value that is not one of the fixed points. Specifically, we have

$$\begin{aligned} a_1 &= \frac{3}{a_0} \\ a_2 &= \frac{3}{a_1} = \frac{3}{\dfrac{3}{a_0}} = a_0 \end{aligned}$$

Thus, $a_3 = a_1$, $a_4 = a_2 = a_0$, and so on.

The last two examples illustrate that fixed points are only candidates for limits and that, depending on the initial condition, the sequence $\{a_n\}$ may or may not converge to a given fixed point. If we know, however, that a sequence $\{a_n\}$ does converge, then the limit of the sequence must be one of the fixed points.

This leaves us with the question of how do we know when a recursive sequence is going to converge. We refer to Theorem 11.12 in the text, the *Monotonic Sequence Theorem*.

Theorem 1.1 (Monotonic Sequence Theorem)**.** *Every bounded, monotonic sequence converges.*

If we are given a sequence in a recursive formula, $a_n = f(a_{n-1})$, we will need to check that it is bounded, check that it is monotonic (increasing or decreasing), and then find any fixed points. We can check if the given function, $f(x)$, is increasing or decreasing by using derivatives, but this does not help us with the monotonicity of the sequence directly.

There are two crucial conditions that must be met to insure the monotonicity of the sequence. Consider the recursive sequence with $a_{n+1} = f(a_n)$ and a_1 is some given value. Solve the equation $f(x) = x$ and assume that we find two fixed points a and b so that $f(a) = a$ and $f(b) = b$.

1. *No fixed point of f is located between a and b.*
   Then, for $a < x < b$, the graph of $y = f(x)$ lies either entirely above or entirely below the graph of $y = x$, the *diagonal*.

2. *The function maps the interval (a, b) to itself.*
   By this we mean that for all $x \in (a, b)$, $a < f(x) < b$.

Based on these two conditions we can see that if the graph of f lies above the line $y = x$, then $a_2 = f(a_1) > a_1$ and the sequence will be increasing and $\lim_{n\to\infty} a_n = b$. If the graph lies below the line $y = x$, then $a_2 = f(a_1) < a_1$ and the sequence will be decreasing and $\lim_{n\to\infty} a_n = a$.

Example 1.8. Consider the recursive sequence $a_{n+1} = \dfrac{5}{6 - a_n}$ with $a_1 = 4$. We want to show that $\{a_n\}$ converges and find its limit.

First, we look at the function: $f(x) = \dfrac{5}{6 - x}$. Note that we have that $a_{n+1} = f(a_n)$ for all $n = 1, 2, 3, \ldots$, so this function generates our sequence.

Step 1 Find the fixed points.

$$\frac{5}{6 - x} = x \Rightarrow 6x - x^2 = 5 \Rightarrow x^2 - 6x + 5 = 0 \Rightarrow x = 1 \text{ or } x = 5.$$

Step 2 Note that for $x > 1$ (which are the only values in which we are interested) we have that $1 \leq f(x) < 6$ on $(1, 5)$. Thus, the function and hence the sequence are bounded.

Step 3 Note that for all $1 < x < 5$ we have that $1 < f(x) < 5$, so the function maps the interval $(1, 5)$ to itself.

Step 4 Note that the initial point a_1 lies between the two fixed points, so we only need to know that the sequence is monotonic on the interval $(1, 5)$, between the fixed points. This is easy since $f'(x) = \frac{5}{(6-x)^2} > 0$ for all $x \neq 6$. Hence, f is increasing. Also, $f''(x) = 10/(6-x)^3$ and the graph is concave up. Thus, the graph must lie below the line $y = x$ and the sequence is decreasing. Thus, since we start at 4, the limit will be 1.

n	1	2	3	4	5	6	7	8	9	10
a_n	4	2.50	1.429	1.094	1.019	1.004	1.001	1.000	1.000	1.000006

1.2 Examples of Recursive Processes

1. Newton's Method: In Calculus I we saw that we could find the root of an equation $f(x) = 0$ by iterating the recursive formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   There is no general formula for the nth term.

2. Compound interest/population growth: The growth of an investment paying i% per annum compounded n times per year from the kth period to the $(k+1)$st period is given by

$$A_{k+1} = A_k \left(1 + \frac{i}{n}\right).$$

   There is a general formula for the kth term.

3. Logistic population growth: $N_{t+1} = N_t[1 + R(1 - N_t/K)]$, where K is the carrying capacity of the environment and R is the growth parameter of the population.

4. Drug concentration: A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug present on the preceding day remains in the bloodstream. If the daily dose raises the concentration by 0.2 mg/mL, the concentration on the next day is $C_{n+1} = 0.3C_n + 0.2$.
   There is a general formula for the nth day.

1.3 Exercises

Check that the following sequences converge and find the limit.

1. $a_{n+1} = \frac{1}{2}(a_n + 8)$, $a_0 = 4$.

2. $a_{n+1} = a_n^2 - 2a_n + 2$, $a_0 = \frac{3}{2}$.

3. $a_{n+1} = -\sqrt{24 - 2a_n}, a_0 = 11.$
4. $a_{n+1} = 2{a_n}^{2/3}, a_0 = 1$
5. $a_{n+1} = 2 - \dfrac{1}{2 + a_n}, a_0 = 5$
6. $a_{n+1} = \dfrac{5a_n}{3 + a_n}, a_0 = 1$
7. $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{9}{a_n}\right), a_0 = 15$

1.4 References

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