MA 114 Worksheet #01: Integration by parts

1. Which of the following integrals should be evaluated using substitution and which should be evaluated using integration by parts?

   (a) $\int x\cos(x^2)\,dx$,

   (b) $\int e^x \sin(x)\,dx$,

   (c) $\int \frac{\ln(\arctan(x))}{1+x^2}dx$,

   (d) $\int xe^{x^2}\,dx$

2. Evaluate the following integrals using integration by parts:

   (a) $\int x^2\sin(x)\,dx$,

   (b) $\int (2x+1)e^x\,dx$,

   (c) $\int x\sin(3-x)\,dx$,

   (d) $\int 2x\arctan(x)\,dx$,

   (e) $\int \ln(x)\,dx$

   (f) $\int x^5\ln(x)\,dx$

   (g) $\int e^x\cos(x)\,dx$

   (h) $\int x\ln(1+x)\,dx$ Hint: Make a substitution first, then try integration by parts.

3. Evaluate the definite integral $\int_0^3 x\sin(3-x)\,dx$.

4. Let $f(x)$ be a twice differentiable function with $f(1)=2$, $f(4)=7$, $f'(1)=5$ and $f'(4)=3$. Evaluate $\int_1^4 xf''(x)\,dx$

5. If $f(0)=g(0)=0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx.$$

MA 114 Worksheet #02: Special Trigonometric Integrals

1. Compute the following integrals:

   (a) $\displaystyle\int \sin(x)\sec^2(x)\,dx$

   (b) $\displaystyle\int \sin^3(x)\,dx$

   (c) $\displaystyle\int_0^{\pi/2} \cos^2(x)\,dx$

   (d) $\displaystyle\int \sqrt{\cos(x)}\sin^3(x)\,dx$

   (e) $\displaystyle\int_0^{2\pi} \sin^2\left(\tfrac{1}{3}\theta\right)\,d\theta$

   (f) $\displaystyle\int_0^{\pi/2} (2-\sin(\theta))^2\,d\theta$

   (g) $\displaystyle\int 4\sin^2(x)\cos^2(x)\,dx$

   (h) $\displaystyle\int \cos^5(x)\,dx.$

2. Find the anti-derivative $\int \cot(x)\,dx$. Hint: Substitute $u=\sin(x)$.

3. Evaluate $\displaystyle\int \sin x\cos x\,dx$ by four methods:

   (a) the substitution $u=\cos(x)$;

   (b) the substitution $u=\sin(x)$;

   (c) the identity $\sin 2x = 2\sin(x)\cos(x)$;

   (d) integration by parts

   Explain the different appearances of the answers.

4. Find the area of the region bounded by the curves $y=\sin^2(x)$ and $y=\sin^3(x)$ for $0\le x\le \pi$.

MA 114 Worksheet #03: Trig Substitution

1. Use the trigonometric substitution $x = \sin(u)$ to find $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx$.

   Remark: This exercise verifies one of the basic anti-derivatives we learned in Calculus I. On an exam, you would be expected to know this anti-derivative and would not be expected to show work to evaluate the anti-derivative by substitution.

2. Compute the following integrals:

   (a) $\displaystyle\int_0^2 \frac{u^3}{\sqrt{16-u^2}}du$

   (b) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}dx$

   (c) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

   (d) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

   (e) $\displaystyle\int \frac{1}{(1+x)^2}\,dx$

   (f) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}dx$

3. Evaluate the following integrals. One may be easily evaluated by substitution $u = 1+x^2$ and for the other use an appropriate trigonometric substitution.

$$\int \frac{\sqrt{1+x^2}}{x}dx \qquad \int \frac{x}{\sqrt{1+x^2}}dx$$

4. (a) Evaluate the integral $\displaystyle\int_0^r \sqrt{r^2-x^2}dx$ using trigonometric substitution.

   (b) Use your answer to part a) to verify the formula for the area of a circle of radius r.

5. Let $r > 0$. Consider the identity

$$\int_0^s \sqrt{r^2-x^2}\,dx = \frac{1}{2}r^2\arcsin\left(\frac{s}{r}\right) + \frac{1}{2}s\sqrt{r^2-s^2}$$

   where $0 \le s \le r$.

   (a) Plot the curves $y = \sqrt{r^2-x^2}$, $x = s$, and $y = \dfrac{x}{s}\sqrt{r^2-s^2}$.

   (b) Using part (a), verify the identity geometrically.

   (c) Verify the identity using trigonometric substitution.

MA 114 Worksheet #04: Integration by Partial Fractions

1. Write out the general form for the partial fraction decomposition but do not determine the numerical value of the coefficients.

   (a) $\dfrac{1}{x^2+3x+2}$

   (b) $\dfrac{x+1}{x^2+4x+4}$

   (c) $\dfrac{x}{(x^2+1)(x+1)(x+2)}$

   (d) $\dfrac{2x+5}{(x^2+1)^3(2x+1)}$

2. Based on your work in the previous question, can you conjecture (guess) a relation between the degree of the denominator of the rational function and the number of coefficients in the partial fraction decomposition?

3. Find the partial fraction decomposition for the following rational functions.

   (a) $\dfrac{1}{x^2-4}$

   (b) $\dfrac{x^3}{x^2-4}$ Hint: This is not a proper rational function, so you will begin by dividing.

   (c) $\dfrac{x}{(x^2+1)(x+1)}$

4. Compute the following integrals.

   (a) $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$

   (b) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

   (c) $\displaystyle\int \frac{x^3-2x^2+1}{x^3-2x^2}\,dx$

   (d) $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$

   (e) $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$

5. Compute

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$$

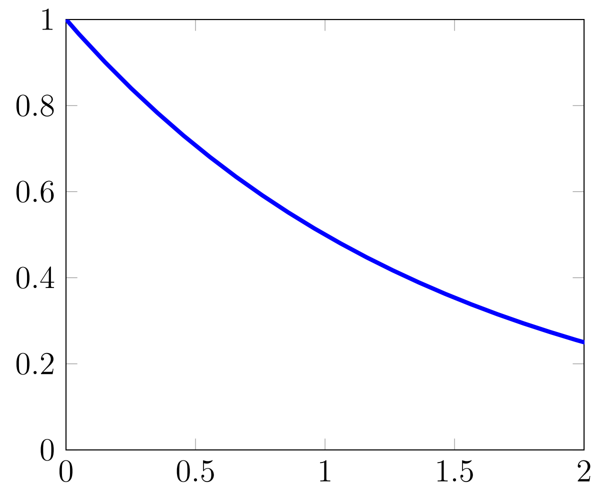
   by first making the substitution $u=\sqrt[6]{x}$.

MA 114 Worksheet #05: Numerical Integration

1. (a) Write down the Midpoint rule and illustrate how it works with a sketch.
   (b) Write down the Trapezoid rule, illustrate how it works with a sketch, and write down the error bound associated with it.
   (c) How large should n be in the Midpoint rule so that you can approximate
   $$\int_0^1 \sin(x)\,dx$$
   with an error less than 10^{-7}?

2. Use the Midpoint rule to approximate the value of $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx$ with $n = 4$. Draw a sketch to determine if the approximation is an overestimate or an underestimate of the integral.

3. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\,dx$, where f is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of sub- intervals were used in each case.
   (a) Which rule produced which estimate?
   (b) Between which two approximations does the true value of $\int_0^2 f(x)\,dx$ lie?



4. Draw the graph of $f(x) = \sin\left(\frac{1}{2}x^2\right)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\,dx$.
   (a) Use the graph to decide whether L_2, R_2, M_2, and T_2 underestimate or overestimate I.
   (b) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.
   (c) Compute L_5, R_5, M_5, and T_5. From the graph, which do you think gives the best estimate of I?

5. The velocity in meters per second for a particle traveling along the axis is given in the table below. Use the Midpoint rule and Trapezoid rule to approximate the total distance the particle traveled from $t = 0$ to $t = 6$.

t	$v(t)$
0	0.75
1	1.34
2	1.5
3	1.9
4	2.5
5	3.2
6	3.0

MA 114 Worksheet #06: Simpson's Rule and Improper Integrals

1. (a) Write down Simpson's rule and illustrate how it works with a sketch.

   (b) How large should n be in Simpson's rule so that you can approximate

   $$\int_0^1 \sin x\, dx$$

   with an error less than 10^{-7}?

2. Approximate the integral

   $$\int_1^2 \frac{1}{x}\, dx$$

   using Simpson's rule. Choose n so that your error is certain to be less than 10^{-3}. Compute the exact value of the integral and compare to your approximation.

3. Simpson's Rule turns out to exactly integrate polynomials of degree three or less. Show that Simpson's rule gives the *exact* value of $\int_0^h p(x)\, dx$ where $h > 0$ and $p(x) = ax^3 + bx^2 + cx + d$. [Hint: First compute the exact value of the integral by direct integration. Then apply Simpson's rule with $n = 2$ and observe that the approximation and the exact value are the same.]

4. For each of the following, determine whether the integral is proper or improper. If it is improper, explain why. Do ***not*** evaluate any of the integrals.

   (a) $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\, dx$

   (b) $\displaystyle\int_1^2 \frac{1}{2x - 1}\, dx$

   (c) $\displaystyle\int_1^2 \ln(x - 1)\, dx$

   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2}\, dx$

   (e) $\displaystyle\int_0^{\pi/2} \sec x\, dx$

5. For the integrals below, determine if the integral is convergent or divergent. Evaluate the convergent integrals.

   (a) $\displaystyle\int_{-\infty}^{0} \frac{1}{2x - 1}\, dx$

   (b) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\, dx$

   (c) $\displaystyle\int_0^2 \frac{x - 3}{2x - 3}\, dx$

   (d) $\displaystyle\int_0^{\infty} \sin\theta\, d\theta$

6. Consider the improper integral
$$\int_1^\infty \frac{1}{x^p}\,dx.$$
Integrate using the generic parameter p to prove the integral converges for $p > 1$ and diverges for $p \leq 1$. You will have to distinguish between the cases when $p = 1$ and $p \neq 1$ when you integrate.

7. Use the Comparison Theorem to determine whether the following integrals are convergent or divergent.
   (a) $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$
   (b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^6+x}}\,dx$

8. Explain why the following computation is wrong and determine the correct answer. (Try sketching or graphing the integrand to see where the problem lies.)
$$\begin{aligned}\int_2^{10} \frac{1}{2x-8}dx &= \frac{1}{2}\int_{-4}^{12} \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|x|\Big|_{-4}^{12} \\ &= \frac{1}{2}(\ln 12 - \ln 4)\end{aligned}$$
where we used the substitution $u(x) = 2x - 8$.

9. A manufacturer of light bulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the companys bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.
   (a) Make a rough sketch of what you think the graph of F might look like.
   (b) What is the meaning of the derivative $r(t) = F'(t)$?
   (c) What is the value of $\int_0^\infty r(t)\,dt$? Why?

MA 114 Worksheet #07: Sequences

1. (a) Give the precise definition of a **sequence**.
   (b) What does it mean to say that $\lim_{x\to a} f(x) = L$ when $a = \infty$? Does this differ from $\lim_{n\to\infty} f(n) = L$? Why or why not?
   (c) What does it means for a sequence to converge? Explain your idea, not just the definition in the book.
   (d) Sequences can diverge in different ways. Describe two distinct ways that a sequence can diverge.
   (e) Give two examples of sequences which converge to 0 and two examples of sequences which converges to a given number $L \neq 0$.

2. Write the first four terms of the sequences with the following general terms:
   (a) $\dfrac{n!}{2^n}$
   (b) $\dfrac{n}{n+1}$
   (c) $(-1)^{n+1}$
   (d) $\{a_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{3}{n}$.
   (e) $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2^{-n} + 2$.
   (f) $\{b_k\}_{k=1}^{\infty}$ where $b_k = \dfrac{(-1)^k)}{k^2}$.

3. Find a formula for the nth term of each sequence.
   (a) $\left\{\dfrac{1}{1}, -\dfrac{1}{8}, \dfrac{1}{27}, -\dfrac{1}{64}, \dots\right\}$
   (b) $\left\{1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dots\right\}$
   (c) $\{1, 0, 1, 0, 1, 0, \dots\}$
   (d) $\left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \dots, \right\}$

4. Suppose that a sequence $\{a_n\}$ is bounded above and below. Does it converge? If not, find a counterexample.

5. The limit laws for sequences are the same as the limit laws for functions. Suppose you have sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} a_n = 15$, $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 1$. Use the limit laws of sequences to answer the following questions.
   (a) Does the sequence $\left\{\dfrac{a_n \cdot c_n}{b_n + 1}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?
   (b) Does the sequence $\left\{\dfrac{a_n + 3 \cdot c_n}{2 \cdot b_n + 2}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?

MA 114 Worksheet #08: Review for Exam 01

1. Find the following antiderivatives

   (a) $\int x^2 \sin(2x)\,dx$

   (b) $\int xe^{2x}\,dx$

   (c) $\int \frac{dx}{x^2+2x+10}$

   (d) $\int \frac{x+3}{(x-6)(x-3)}\,dx$

   (e) $\int \frac{3x+6}{x^2-10x+24}\,dx$

   (f) $\int \frac{3x^2+9x+8}{x^2(x+2)^2}\,dx$

   (g) $\int \sin^5(x)\cos(x)\,dx$

   (h) $\int \frac{dx}{x\sqrt{x^2+9}}$

   (i) $\int \sqrt{16+4x^2}\,dx$

   (j) $\int x^3\sqrt{9-x^2}\,dx$

   (k) $\int_1^2 \frac{dx}{x\ln x}$

2. Evaluate the following integrals.

   (a) $\int_0^{\pi} \sin^2(x)\,dx$

   (b) $\int_1^{\infty} xe^{-2x}\,dx$

3. Let $f(x) = e^{-x^2}$. Find a value of N for use in the trapezoid rule to compute

$$\int_0^3 e^{-x^2}\,dx$$

   accurate to within 0.0001. Hint: $|f(x)| \leq 1$ and $|f'(x)| \leq 1$ on $[0,3]$.

4. For which values of p does the improper integral

$$\int_0^{\infty} \frac{dx}{(1+x)^p}$$

   converge? If it converges, to what value does it converge?

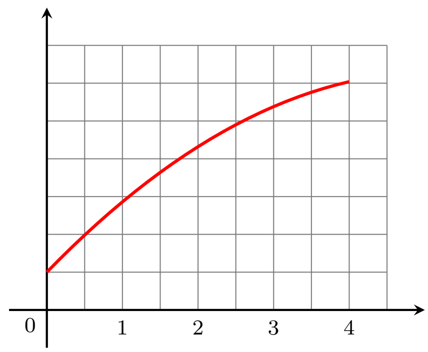
5. Determine if each of the following integrals is convergent and evaluate the ones that are convergent.

   (a) $\int_{-1}^{1} \frac{1}{x}\,dx$

   (b) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

6. Calculate M_6 and T_6 to approximate $\int_{-2}^{1} e^{x^2}\,dx$.

7. Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is shown below. For any value of n, list the numbers L_n, R_n, M_n, and T_n in increasing order.



8. An airplane's velocity is recorded at 5-minute intervals during a 1 hour period with the following results, in miles per hour:

$$\begin{array}{ccccccc} 550, & 575, & 600, & 580, & 610, & 640, & 625, \\ 595, & 590, & 620, & 640, & 640, & 630 & \end{array}$$

(a) Use Simpson's Rule to estimate the distance traveled during the hour.

(b) Use the trapezoid rule to estimate the distance traveled during the hour.

MA 114 Worksheet #09: Recursive Sequences and Series

1. Write out the first five terms of
   (a) $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 3a_{n-1} + {a_n}^2$.
   (b) $a_1 = 6$, $a_{n+1} = \dfrac{a_n}{n}$.
   (c) $a_1 = 2$, $a_{n+1} = \dfrac{a_n}{a_n + 1}$.
   (d) $a_1 = 1$, $a_{n+1} = \sqrt{\left(\dfrac{2}{a_n}\right)^2 + 1}$.
   (e) $a_1 = 2$, $a_2 = 1$, and $a_{n+1} = a_n - a_{n-1}$.

2. (a) For what values of x does the sequence $\{x^n\}_{n=1}^{\infty}$ converge?
   (b) For what values of x does the sequence $\{n^x\}_{n=1}^{\infty}$ converge?
   (c) If $\lim\limits_{n\to\infty} b_n = \sqrt{2}$, find $\lim\limits_{n\to\infty} b_{n-3}$.

3. (a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n > 1.$$

   (b) What happens if the first term is $a_1 = 2$?

4. For each of the recursively defined sequences, i) find the first three terms, ii) find a formula for a_n in terms of n, iii) check your answer by comparing the values given by the formula with your answer the first step.
   (a) $a_1 = 7$ and $a_n = a_{n-1} + 2$
   (b) $a_1 = 3$ and $a_n = 4 \cdot a_{n-1}$
   (c) $a_1 = 5$ and $a_n = -2 \cdot a_{n-1}$
   (d) $a_1 = 1$ and $a_n = n \cdot a_{n-1}$

5. Suppose that the sequence $a_1, a_2, a_3, \ldots$ satisfies the recursion relation

$$a_n = \frac{1}{2}(a_{n-1} + \frac{3}{a_{n-1}}), \qquad n \geq 2,$$

   and $a = \lim_{n\to\infty} a_n$ exists. Find a.

6. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.
   (a) Show that the catfish population P_n after n months is given recursively by

$$P_n = 1.08P_{n-1} - 300 \qquad P_0 = 5000.$$

   (b) How many catfish are in the pond after six months?

MA 114 Worksheet #10: Series and The Integral Test

1. Identify the following statements as true or false and explain your answers.
   (a) If the sequence of partial sums of an infinite series is bounded the series converges.
   (b) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} a_n$ if the series converges.
   (c) $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$ if both series converge.
   (d) If c is a nonzero constant and if $\sum_{n=1}^{\infty} ca_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
   (e) A finite number of terms of an infinite series may be changed without affecting whether or not the series converges.
   (f) Every infinite series with only finitely many nonzero terms converges.

2. Write the following in summation notation:
   (a) $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots$
   (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

3. Calculate S_3, S_4, and S_5 and then find the sum of the telescoping series $S = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$.

4. Use the formula for the sum of a geometric series to find the sum or state that the series diverges and why:
   (a) $\frac{1}{1} + \frac{1}{8} + \frac{1}{8^2} + \dots$
   (b) $\sum_{n=0}^{\infty} \left(\frac{\pi}{e} \right)^n$

5. Use the Integral Test to determine if the following series converge or diverge:
   (a) $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$
   (b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$
   (c) $\sum_{n=2}^{\infty} \frac{n}{(n^2+2)^{3/2}}$

6. Show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise by the Integral Test.

MA 114 Worksheet #11: Comparison and Limit Comparison Tests

1. (a) Explain the test for divergence. Why should you never use this test to prove that a series converges?
   (b) State the comparison test for series. Explain the idea behind this test.
   (c) Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ satisfy $0 \le x_n \le y_n$ for all n and that $\sum_{n=1}^{\infty} y_n$ is convergent. What can you conclude? What can you conclude if instead $\sum_{n=1}^{\infty} y_n$ diverges?
   (d) State the limit comparison test. Explain how you apply this test.

2. Use the appropriate test — Divergence Test, Comparison Test or Limit Comparison Test — to determine whether the infinite series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{2+5^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+2}{3^n+1}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{10}{n}\right)^{10}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{3n^2+14n+7}$

   (h) $\displaystyle\sum_{n=0}^{\infty} \frac{1+2^n}{2+5^n}$

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+5n+2}$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

   (k) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2-\cos^2 n}$

   (l) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^4}$

   (m) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n+1)!}$

MA 114 Worksheet #12: Alternating Series, Absolute Convergence, & Conditional Convergence

1. (a) Let $a_n = \dfrac{n}{3n+1}$. Does $\{a_n\}$ converge? Does $\sum_{n=1}^{\infty} a_n$ converge?
   (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ where $\lim\limits_{n\to\infty} a_n = 0$.
   (c) Does there exist a convergent series $\sum_{n=1}^{\infty} a_n$ which satisfies $\lim\limits_{n\to\infty} a_n \neq 0$? Explain.
   (d) When does a series converge absolutely? When does a series converge conditionally?
   (e) State the alternating series test.
   (f) State the Alternating Series Estimation Theorem.

2. Prove that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

3. Test the following series for convergence or divergence.
   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2n}$
   (b) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{2/3}}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n+5^n}$
   (e) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln(n)}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-5}{18}\right)$

4. Use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$
   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

5. Approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!}$ correct to four decimal places; *i.e.*, so that $|\text{error}| < 0.00005$.

MA 114 Worksheet #13: The Ratio and Root Tests

1. (a) State the Root Test.
   (b) State the Ratio Test.

2. Identify the following statements as true or false. If the statement is true, cite evidence from the text to support it. If the statement is false, correct it so that it is a true statement from the text.
   (a) To prove that the series $\sum_{n=1}^{\infty} a_n$ converges you should compute the limit $\lim_{n\to\infty} a_n$. If this limit is 0 then the series converges.
   (b) To apply the Ratio Test to the series $\sum_{n=1}^{\infty} a_n$ you should compute $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If this limit is less than 1 then the series converges absolutely.
   (c) To apply the Root Test to the series $\sum_{n=1}^{\infty} a_n$ you should compute $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If this limit is 1 or larger then the series diverges.
   (d) One way to prove that a series is convergent is to prove that it is absolutely convergent.
   (e) An infinite series converges when the sequence of partial sums converges.

3. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Remember that you may use **any** tests you have learned.

   (a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n^3+2n}{4n^3+1}\right)^n$
   (b) $\displaystyle\sum_{n=1}^{\infty} 13\cos(5)^{n-1}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(11-\cos^2(n))^n}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$
   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

MA 114 Worksheet #14: Power Series

1. (a) Give the definition of the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$

   (b) For what values of x does the series $\sum_{n=1}^{\infty} 2\big(\cos(x)\big)^{n-1}$ converge?

   (c) Find a formula for the coefficients c_k of the power series $\frac{1}{0!}+\frac{2}{1!}x+\frac{3}{2!}x^2+\frac{4}{3!}x^3+\cdots$.

   (d) Find a formula for the coefficients c_n of the power series $1+2x+x^2+2x^3+x^4+2x^5+x^6+\cdots$.

   (e) Suppose $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$ where $c \neq 0$. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$.

   (f) Consider the function $f(x) = \frac{5}{1-x}$. Find a power series that is equal to $f(x)$ for every x satisfying $|x| < 1$.

   (g) Define the terms *power series*, *radius of convergence*, and *interval of convergence.*

2. Find the radius and interval of convergence for

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$

   (b) $4\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$

   (c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$

   (d) $\sum_{n=0}^{\infty} n!(x-2)^n$

   (e) $\sum_{n=0}^{\infty} (5x)^n$

   (f) $\sum_{n=0}^{\infty} \sqrt{n}x^n$

   (g) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

   (h) $\sum_{n=3}^{\infty} \frac{x^n}{3^n \ln n}$

   (i) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

   (j) $\sum_{n=4}^{\infty} \frac{(-1)^n x^n}{n^4}$

   (k) $\sum_{n=3}^{\infty} \frac{(5x)^n}{n^3}$

3. Use term-by-term integration and the fact that $\int_0^x \frac{1}{1+t^2}\,dt = \arctan(x)$ to derive a power series centered at $x=0$ for the arctangent function. HINT: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$.

4. Use the same idea as above to give a series expression for $\ln(1+x)$, given that $\int_0^x \frac{dt}{1+t} = \ln(1+x)$. You will again want to manipulate the fraction $\frac{1}{1+x} = \frac{1}{1-(-x)}$ as above.

5. Write $(1+x^2)^{-2}$ as a power series. HINT: use term-by-term differentiation.

MA 114 Worksheet #15: Taylor and Maclaurin Series

1. (a) Suppose that $f(x)$ has a power series representation for $|x| < R$. What is the general formula for the Maclaurin series for f?
   (b) Suppose that $f(x)$ has a power series representation for $|x - a| < R$. What is the general formula for the Taylor series for f about a?
   (c) Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$. Find the Maclaurin series for f.
   (d) Let $f(x) = 1 + 2x + 3x^2 + 4x^3$. Find the Taylor series for $f(x)$ centered at $x = 1$.

2. Assume that each of the following functions has a power series expansion. Find the Maclaurin series for each. Be sure to provide the domain on which the expansion is valid.
   (a) $f(x) = \ln(1 + x)$
   (b) $f(x) = xe^{2x}$

3. Use a known Maclaurin series to obtain the Maclaurin series for the given function. Specify the radius of convergence for the series.
   (a) $f(x) = \dfrac{x^2}{1 - 3x}$
   (b) $f(x) = e^x + e^{-x}$
   (c) $f(x) = e^{-x^2}$
   (d) $f(x) = x^5 \sin(3x^2)$
   (e) $f(x) = \sin^2 x$.
   Hint: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

4. Find the following Taylor expansions about $x = a$ for each of the following functions and their associated radii of convergence.
   (a) $f(x) = e^{5x}$, $a = 0$.
   (b) $f(x) = \sin(\pi x)$, $a = 1$.

5. Differentiate the series
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
   to find a Taylor series for $\cos(x)$.

6. Use Maclaurin series to find the following limit: $\displaystyle\lim_{x\to 0} \frac{x - \tan^{-1}(x)}{x^3}$.

7. Approximate the following integral using a 6th order polynomial for $\cos(x)$.
$$\int_0^1 x\cos(x^3)\, dx$$

8. Use power series multiplication to find the first three terms of the Maclaurin series for
$$f(x) = e^x \ln(1 - x).$$

MA 114 Worksheet #16: Review for Exam 2

1. List the first five terms of the sequence:

   (a) $a_n = \dfrac{(-1)^n n}{n! + 1}$

   (b) $a_1 = 6$, $a_{n+1} = \dfrac{a_n}{n}$.

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = 3^n 7^{-n}$

   (b) $a_n = \dfrac{(-1)^{n+1} n}{n + \sqrt{n}}$

   (c) $a_n = \dfrac{\ln n}{\ln 2n}$

   (d) $a_n = \dfrac{\cos^2(n)}{2^n}$

3. Explain what it means to say that $\displaystyle\sum_{n=1}^{\infty} a_n = 2$.

4. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{3^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{n-1}}{3^n}$

5. Determine whether the given series converges or diverges and state which test you used.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{7\sqrt{n}}{5n^{3/2} + 3n - 2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} n! e^{-8n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln(n)}{5n+7}\right)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{9n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \arctan(n)$

6. Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

7. Find the radius and interval of convergence of each power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{4^n n^4}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

8. Find a power series representation for each function and determine its radius of convergence.

   (a) $f(x) = \dfrac{5}{1-4x^2}$

   (b) $f(x) = \dfrac{x^2}{x^4+16}$

   (c) $f(x) = \dfrac{3}{2+2x}$

   (d) $f(x) = e^{-x^2}$

9. Using the formula
$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt$$
find a power series for $\ln(1+x)$ and state its radius of convergence.

10. Use the Maclaurin series for $\cos(x)$ to compute
$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2}.$$

MA 114 Worksheet #17: Average value of a function

1. Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.

2. Find the average value of the following functions over the given interval.

   (a) $f(x) = x^3,\ [0, 4]$

   (b) $f(x) = x^3,\ [-1, 1]$

   (c) $f(x) = \cos(x),\ \left[0, \dfrac{\pi}{6}\right]$

   (d) $f(x) = \dfrac{1}{x^2+1},\ [-1, 1]$

   (e) $f(x) = \dfrac{\sin(\pi/x)}{x^2},\ [1, 2]$

   (f) $f(x) = e^{-nx},\ [-1, 1]$

   (g) $f(x) = 2x^3 - 6x^2,\ [-1, 3]$

   (h) $f(x) = x^n$ for $n \geq 0$, $[0, 1]$

3. In a certain city the temperature (in $^\circ F$) t hours after 9 am was modeled by the function $T(t) = 50 + 14\sin\left(\dfrac{\pi t}{12}\right)$. Find the average temperature during the period from 9 am to 9 pm.

4. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

   where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity (with respect to r) over the interval $0 < r < R$. Compare the average velocity with the maximum velocity.

5. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t. Then use this new function to compute the average volume of inhaled air in the lungs in one respiratory cycle.

MA 114 Worksheet #18: Volumes I

1. If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?

2. Calculate the volume of the solid: the base is a square, one of whose sides is the interval $[0, l]$ along the x-axis, and the cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

3. Calculate the volume of the solid whose base is the region enclosed by $y = x^2$ and $y = 3$, where cross sections perpendicular to the y-axis are squares.

4. The base of a certain solid is the triangle with vertices at $(10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.

5. Calculate the volume of the solid whose base is a circle of radius r centered at the origin and which has square cross sections perpendicular to the x-axis.

6. Calculate the volume of the following solid. The base is the parabolic region $\{(x, y) \mid x^2 \leq y \leq 4\}$. The cross sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse lies in the region.

7. Sketch the solid whose volume is given by the integral
$$\pi \int_0^1 (y^2 + 1)^2 - 1 \, dy.$$

8. In parts a) and b), use disks or washers to find the an integral expression for the volume of the solid obtained by rotating the given region about the specified line. Evaluate the integrals the integrals to find the volume.
   (a) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the x-axis.
   (b) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.
   (c) Compare the volumes you found in parts (a) and (b). Which is bigger? Why?

9. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

10. The torus is the solid obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$.
[Hint: Draw a picture, set up the problem and evaluate the integral by interpreting it as the area of a circle.]

MA 114 Worksheet #19: Volumes II

1. (a) Write a general integral to compute the volume of a solid obtained by rotating the region under $y = f(x)$ over the interval $[a, b]$ about the y-axis using the method of cylindrical shells.

   (b) If you use the disk method to compute the same volume, are you integrating with respect to x or y? Why?

2. Sketch the enclosed region and use the shell method to calculate the volume of the solid obtained by rotating the region about the y-axis.

   (a) $y = 3x - 2$, $y = 6 - x$, $x = 0$

   (b) $y = x^2$, $y = 8 - x^2$, $x = 0$, for $x \geq 0$

   (c) $y = 8 - x^3$, $y = 8 - 4x$, for $x \geq 0$

3. For each of the following, sketch the region R and write an integral to give the volume of the solid obtained by rotating R about the specified line.

   (a) R is region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$; about the line $y = 2$.

   (b) R is the region bounded by $y = 2 - x^2$ and $y = 1$; about the line $y = -1$.

   (c) R is region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

4. Consider the region R which is enclosed by the curves $y = 2x$ and $y = x^2$. The region R is rotated about the x-axis to obtain a solid S.

   (a) Express the volume of S as an integral using the method of washers.

   (b) Express the volume of S as an integral using the method of shells.

   (c) Evaluate the integrals you found in parts a) and b). Check your work by comparing your answers.

5. A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \leq x \leq 1$ about the y-axis. Soda is extracted from the glass through a straw at the rate of $1/2$ cubic inch per second. How fast is the soda level in the glass dropping when the level is 2 inches? (Answer should be implicitly in units of inches per second.)

MA 114 Worksheet #20: Arc length and surface area

1. (a) Write down the formula for the arc length of a function $f(x)$ over the interval $[a, b]$ including the required conditions on $f(x)$.

   (b) Write down the formula for the surface area of a solid of revolution generated by rotating a function $f(x)$ over the interval $[a, b]$ around the x-axis. Include the required conditions on $f(x)$.

2. Find an integral expression for the arc length of the following curves. Do ***not*** evaluate the integrals.

   (a) $f(x) = \sin(x)$ from $x = 0$ to $x = 2$.

   (b) $f(x) = x^4$ from $x = 2$ to $x = 6$.

   (c) $x^2 + y^2 = 1$

3. Find the arc length of the following curves.

   (a) $f(x) = x^{3/2}$ from $x = 100$ to $x = 101$.

   (b) $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = \pi/3$.

   (c) $f(x) = e^x$ from $x = 0$ to $x = 1$.

4. Set up a function $s(t)$ that gives the arc length of the curve $f(x) = 2x + 1$ from $x = 0$ to $x = t$. Find $s(4)$.

5. Compute the surface areas of revolution about the x-axis over the given interval for the following functions.

   (a) $y = x$, $[0, 4]$

   (b) $y = x^3$, $[0, 2]$

   (c) $y = \left(4 - x^{2/3}\right)^{3/2}$, $[0, 8]$

   (d) $y = e^{-x}$, $[0, 1]$

   (e) $y = \sin x$, $[0, \pi]$

   (f) Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = r^2$ about the x-axis.

   (g) Show that the surface area of a right circular cone of radius r and height h is $\pi r\sqrt{r^2 + h^2}$.
   Hint: Rotate a line $y = mx$ about the x-axis for $0 \leq x \leq h$, where m is determined by the radius r.

MA 114 Worksheet #21: Centers of mass and moments

1. Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates $(1, 2)$, $(-3, 2)$, $(2, -1)$, and $(4, 0)$.
2. Point masses of equal size are placed at the vertices of the triangle with coordinates $(3, 0)$, $(b, 0)$, and $(0, 6)$, where $b > 3$. Find the center of mass.
3. Find the centroid of the region under the graph of $y = 1 - x^2$ for $0 \le x \le 1$.
4. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \le x \le 4$.
5. Find the centroid of the region between $f(x) = x - 1$ and $g(x) = 2 - x$ for $3/2 \le x \le 2$.

MA 114 Worksheet #22: Parametric Curves

1. (a) How is a curve different from a parametrization of the curve?

   (b) Suppose a curve is parameterized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?

   (c) What parametric equations represent the circle of radius 5 with center $(2, 4)$?

   (d) Represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{c^2} = 1$ with parametric equations.

   (e) Do the two sets of parametric equations

   $$y_1(t) = 5\sin(t),\ x_1(t) = 5\cos(t),\ 0 \le t \le 2\pi$$

   and

   $$y_2(t) = 5\sin(t),\ x_2(t) = 5\cos(t),\ 0 \le t \le 20\pi$$

   represent the same parametric curve? Discuss.

2. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \le t \le 2\pi$.

   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.

   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?

   (c) Use the above information to plot the curve.

3. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.

   (a) $x = \sqrt{t}$, $y = 1 - t$ for $t \ge 0$.

   (b) $x = 3t - 5$, $y = 2t + 1$ for $t \in \mathbf{R}$.

   (c) $x = \cos(t)$, $y = \sin(t)$ for $t \in [0, 2\pi]$.

4. Represent each of the following curves as parametric equations traced just once on the indicated interval.

   (a) $y = x^3$ from $x = 0$ to $x = 2$.

   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

5. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

6. For the following parametric curves, find an equation for the tangent to the curve at the specified value of the parameter.
   (a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$.
   (b) $x = \cos(\theta) + \sin(2\theta)$, $y = \cos(\theta)$, at $\theta = \pi/2$.
7. For the following parametric curve, find dy/dx.
   (a) $x = e^{\sqrt{t}}$, $y = t + e^{-t}$.
   (b) $x = t^3 - 12t$, $y = t^2 - 1$.
   (c) $x = 4\cos(t)$, $y = \sin(2t)$.
8. Find d^2y/dx^2 for the curve $x = 7 + t^2 + e^t$, $y = \cos(t) + \dfrac{1}{t}$, $0 < t \leq \pi$.
9. Find the arc length of the following curves.
   (a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.
   (b) $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \leq t \leq 2\pi$.
   (c) $x = 3t^2$, $y = 4t^3$, $1 \leq t \leq 3$.
10. Suppose you wrap a string around a circle. If you unwind the string from the circle while holding it taut, the end of the string traces out a curve called the *involute* of the circle. Suppose you have a circle of radius r centered at the origin, with the end of the string all the way wrapped up resting at the point $(r, 0)$. As you unwrap the string, define θ to be the angle formed by the x-axis and the line segment from the center of the circle to the point up to which you have unwrapped the string.
    (a) Draw a picture and label θ.
    (b) Show that the parametric equations of the involute are given by $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$. Hint: For any angle θ, the line segment between the point up to which you have unwrapped the string and the point at the end of the string has the same length as the arc of the circle corresponding to the angle θ.
    (c) Find the length of the involute for $0 \leq \theta \leq 2\pi$.

MA 114 Worksheet #23: Polar coordinates

1. Convert from rectangular to polar coordinates:
   (a) $(1, \sqrt{3})$
   (b) $(-1, 0)$
   (c) $(2, -2)$
2. Convert from polar to rectangular coordinates:
   (a) $\left(2, \frac{\pi}{6}\right)$
   (b) $\left(-1, \frac{\pi}{2}\right)$
   (c) $\left(1, -\frac{\pi}{4}\right)$
3. List all the possible polar coordinates for the point whose polar coordinates are $(-2, \pi/2)$.
4. Sketch the graph of the polar curves:
   (a) $\theta = \frac{3\pi}{4}$
   (b) $r = \pi$
   (c) $r = \cos\theta$
   (d) $r = \cos(2\theta)$
   (e) $r = 1 + \cos\theta$
   (f) $r = 2 - 5\sin\theta$
5. Find the equation in polar coordinates of the line through the origin with slope $\frac{1}{3}$.
6. Find the polar equation for:
   (a) $x^2 + y^2 = 9$
   (b) $x = 4$
   (c) $y = 4$
   (d) $xy = 4$
7. Convert the equation of the circle $r = 2\sin\theta$ to rectangular coordinates and find the center and radius of the circle.
8. Find the distance between the polar points $(3, \pi/3)$ and $(6, 7\pi/6)$.

MA 114 Worksheet #24: Review for Exam 03

1. Find the volume of the following solids.
   (a) The solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ about the x-axis,
   (b) The solid obtained by rotating the region bounded by $x = y^2$ and $x = 1$ about the line $x = 1$,
   (c) The solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about the line $x = 1$,
   (d) The solid with circular base of radius 1 and cross-sections perpendicular to the base that are equilateral triangles.
2. Find the area of the surface of revolution obtained by rotating the given curve about the given axis.
   (a) $y = \sqrt{x+1},\ 0 \le x \le 3$; about x-axis,
   (b) $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5$; about y-axis.
3. Compute the arc length of the following curves.
   (a) $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le 2\pi$,
   (b) $y = \sqrt{2 - x^2},\ 0 \le x \le 1$.
4. Find the centroid of the region bounded by $y = \sqrt{x}$ and $y = x$.
5. Find the average value of the function $y = 3\sin(x) + \cos(2x)$ on the interval $[0, \pi]$.
6. Compute the slope of the tangent line to the curve in Problem 3(a) above, with $a = 8$, at the point $(1, 3^{3/2})$. Use this to determine an equation for the tangent line.
7. Consider the curve given by the parametric equations $(x(t), y(t)) = (t^2, 2t + 1)$.
   (a) Find the tangent line to the curve at $(4, -3)$. Put your answer in the form $y = mx + b$.
   (b) Find second derivative $\dfrac{d^2y}{dx^2}$ at $(x, y) = (4, -3)$. Is the curve concave up or concave down near this point?

MA 114 Worksheet #25: Calculus with polar coordinates

1. Find dy/dx for the following polar curves.

   (a) $r = 2\cos\theta + 1$ (b) $r = 1/\theta$ (c) $r = 2e^{-\theta}$

2. In each of the following, compute the slope of the tangent line at the given point. Then sketch the curve and the tangent line.

   (a) $r = \sin\theta$ at $\theta = \pi/3$. (b) $r = 1/\theta$ at $\theta = \pi/2$.

3. (a) Give the formula for the area of region bounded by the polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$. Give a geometric explanation of this formula.

   (b) Give the formula for the length of the polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$.

   (c) Use these formulas to establish the formulas for the area and circumference of a circle.

4. Find the slope of the tangent line to the polar curve $r = \theta^2$ at $\theta = \pi$.

5. Find the point(s) where the tangent line to the polar curve $r = 2 + \sin\theta$ is horizontal.

6. Find the area enclosed by one leaf of the curve $r = \sin 2\theta$.

7. Find the arc length of the curve $r = 5^\theta$ for $\theta = 0$ to $\theta = 2\pi$.

8. Find the area of the region bounded by $r = \cos\theta$ for $\theta = 0$ to $\theta = \pi/4$.

9. Find the area of the region that lies inside both the curves $r = \sqrt{3}\sin\theta$ and $r = \cos\theta$.

10. Find the area in the first quadrant that lies inside the curve $r = 2\cos\theta$ and outside the curve $r = 1$.

11. Find the length of the curve $r = \theta^2$ for $0 \leq \theta \leq 2\pi$.

12. Write down an integral expression for the length of the curve $r = \sin\theta + \theta$ for $0 \leq \theta \leq \pi$ but do not compute the integral.

MA 114 Worksheet #26: Conic Sections

1. The point in a lunar orbit nearest the surface of the moon is called perilune and the point farthest from the surface is called apolune. The Apollo 11 spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

2. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

3. Use parametric equations and Simpson's Rule with $n = 4$ to estimate the circumference of the ellipse $9x^2 + 4y^2 = 36$. Hint: Find the arc-length of the portion of the ellipse that lies in the first quadrant using Simpson's rule and then multiple by four to obtain the total circumference.

4. Find the area of the region enclosed by the hyperbola $4x^2 - 25y^2 = 100$ and the vertical line through a focus.

5. If an ellipse is rotated about its major axis, find the volume of the resulting solid.

6. Find the centroid of the region enclosed by the x-axis and the top half of the ellipse $9x^2 + 4y^2 = 36$.

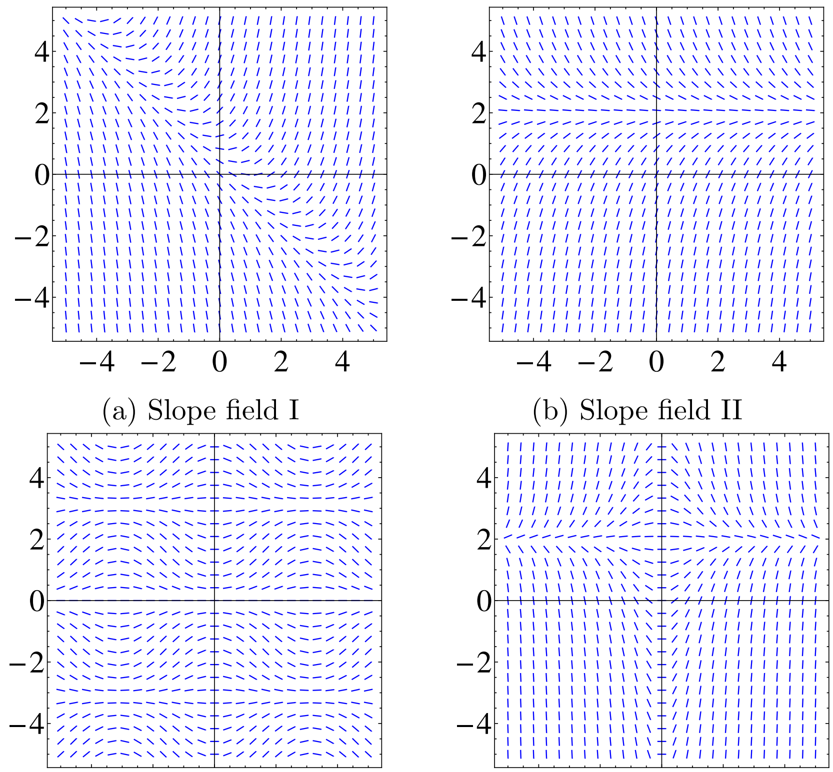
MA 114 Worksheet #27: Differential equations

1. (a) Is $y = \sin(3x) + 2e^{4x}$ a solution to the differential equation $y'' + 9y = 50e^{4x}$? Explain why or why not.
   (b) Explain why every solution of $dy/dx = y^2 + 6$ must be an increasing function.
   (c) What does is mean to say that a differential equation is linear or nonlinear?
2. Find all values of α so that $y(x) = e^{\alpha x}$ is a solution of the differential equation $y'' + y' - 12y = 0$.
3. Find the solution of the initial value problem
$$y(0) = 2, \qquad y' = 3 - y.$$
4. Find the solution of the initial value problem
$$y(0) = -2, \qquad y' = y^2.$$
5. Consider a solution of the differential equation $y' = 3y - 2$. For which values of y is the solution increasing? For which values of y is the solution decreasing?
6. A tank has pure water flowing into it at 10 liters/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liters/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 liters of water. Formulate an initial value problem (that is, a differential equation along with initial conditions) whose solution is the quantity of salt in the tank at any time t. Do not solve the initial value problem.
7. Consider a tank with 200 liters of salt-water solution. A salt-water solution, with a concentration of 2 grams per liter, pours into the tank at a rate of 4 liters per minute. The well-mixed solution in the tank pours out at the same rate of 4 liters/minute. Write a differential equation expressing the rate of change in the concentration, $c(t)$, of salt in the tank. Do not solve.

MA 114 Worksheet #28: Direction fields, Separable differential equations

1. Match the differential equation with its slope field. Give reasons for your answer.

$$y' = 2 - y \qquad y' = x(2-y) \qquad y' = x + y - 1 \qquad y' = \sin(x)\sin(y)$$



(a) Slope field I

(b) Slope field II

(c) Slope Field III

(d) Slope field IV

Figure 1: Slope fields for Problem 1

2. Use slope field labeled IV to sketch the graphs of the solutions that satisfy the given initial conditions.

$$y(0) = -1, \quad y(0) = 0, \quad y(0) = 1.$$

3. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point.

   (a) $y' = y^2$, $(1, 1)$

   (b) $y' = y - 2x$, $(1, 0)$

   (c) $y' = xy - x^2$, $(0, 1)$

4. Consider the autonomous (depends only on y and its derivatives) differential equation $y' = y^2(3-y)(y+1)$. Without solving the differential equation, determine the value of $\lim_{t\to\infty} y(t)$, where the initial value is
   (a) $y(0) = 1$,
   (b) $y(0) = 4$,
   (c) $y(0) = -4$.

5. Use Euler's method with step size 0.5 to compute the approximate y-values, y_1, y_2, y_3, and y_4 of the solution of the initial-value problem $y' = y - 2x$, $y(1) = 0$.

6. Use separation of variables to find the general solutions to the following differential equations.
   (a) $y' + 4xy^2 = 0$
   (b) $\sqrt{1-x^2}y' = xy$
   (c) $(1+x^2)y' = x^3y$
   (d) $y' = 3y - y^2$

MA 114 Worksheet #29: Review for Exam 04

This review worksheet covers only material discussed since Exam III.

To review for your final exam, be sure to study the material from Exams I, II, and III and the review sheets for these exams.

1. Identify and graph the conic section given by each of the following equations. Where applicable, find the foci.
   (a) $x^2 = 4y - 2y^2$
   (b) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x \sin x}{y}, \quad y(0) = -1$$

   It will help to know that

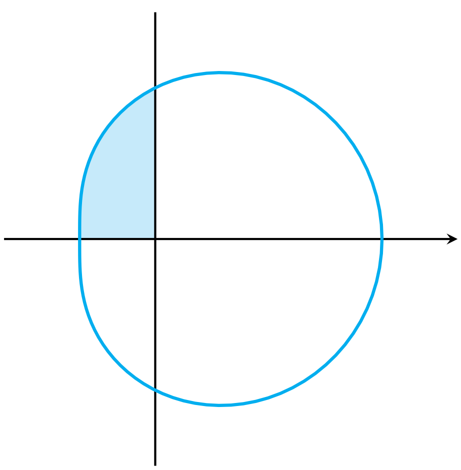
$$\int x \sin x \, dx = \sin x - x \cos x + C$$

3. By converting to Cartesian coordinates, identify and graph the curve $r^2 \sin 2\theta = 1$ (It may help to remember the identity $\sin 2\theta = 2 \sin\theta \cos\theta$).

4. Draw a direction field for the differential equation $y' = y(1-y)$. What are the equilibria? Classify each as stable or unstable.

5. Find the slope of the tangent line to the curve $r = 2\cos\theta$ at $\theta = \pi/3$.

6. Find the area of the region shown.



$r = 2 + \cos\theta$

7. Find the exact length of the polar curve $r = \theta^2$ for $0 \leq \theta \leq 2\pi$.

8. Use Euler's method with step size 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution of the initial-value problem $y' = y + xy$, $y(0) = 1$.

9. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem $y' = x^2y - \frac{1}{2}y^2$, $y(0) = 1$.

10. Solve the following differential equations.

    (a) $\dfrac{dy}{dx} = 3x^2y^2$

    (b) $xyy' = x^2 + 1$

    (c) $\dfrac{dy}{dx} + e^{x+y} = 0$

11. (a) Solve the differential equation $y' = 2x\sqrt{1 - y^2}$.

    (b) Solve the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 0$, and graph the solution.

    (c) Does the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$, have a solution? Explain.