Algebra and Geometry Review (pages 4-5), Solutions

We will make use of the Pythagorean Theorem repeatedly throughout the semester. We will also occasionally use facts about similar triangles. The following problems preview some of the ways we will use these geometric facts.

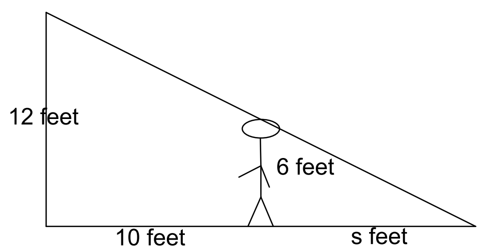
1. A man is six feet tall. The man stands a distance of x feet from a lightpost. The top of the light is 12 feet from the ground.

(a) How long is the man's shadow when the man is 10 feet from the lightpost?

(b) How long is the man's shadow when the man is x feet from the lightpost? Your answer will be a function of x.

(c) How far is the man from the lightpost when his shadow is 15 feet long?

Solution:



(a) Let s denote the length of the man's shadow. Notice that the distance between the lightpost and the end of the man's shadow is then $10 + s$ feet. Due to similar triangles, we have

$$\frac{10+s}{s} = \frac{12}{6} \Longrightarrow 10 + s = 2s$$

and so $s = 10$. i.e., the man's shadow is 10 feet long.

(b) Let x denote the distance between the man and the lightpost and let s denote the length of the man's shadow. Note that the distance between the lightpost and the end of the man's shadow is $x + s$. Due to similar triangles, we have

$$\frac{x+s}{s} = \frac{12}{6} \Longrightarrow x + s = 2s$$

and so $x = s$. i.e., the length of the man's shadow is the same as the distance between the lightpost and the man. (Will these two length's always be equal? What if the man was 5 feet tall? What if the lightpost was 15 feet tall?)

(c) We want $s = 15$, but from the previous part we know $s = x$, so the man must be 15 feet from the lightpost in order for his shadow to be 15 feet long.

2. A ladder is 13 feet long. One end of the ladder is on the ground and the other end rests along a vertical wall.

(a) How far is the base of the ladder from the wall when the top of the ladder is 12 feet from the ground?

(b) How far is the base of the ladder from the wall when the top of the ladder is y feet from the ground?

(c) Let $f(y)$ denote the distance between the wall and the base of the ladder when the top of the ladder is y feet from the ground. Find the domain of $f(y)$. What is the physical meaning of this domain?

Solution: The ladder is the hypotenuse of a right triangle. In part (a), we have a right triangle where one leg is 12 feet long and the hypotenuse is 13 feet. Letting x denote the length of the unknown leg, we have, by the Pythagorean Theorem,

$$x^2 + 12^2 = 13^2 \Longrightarrow x^2 = 169 - 144 = 25$$

so $x = \sqrt{25} = 5$ feet long. In part (b), one leg of the triangle is y feet long and so

$$x^2 + y^2 = 169 \Longrightarrow x = \sqrt{169 - y^2}$$

(c) $f(y) = \sqrt{169 - y^2}$. The inside of the square root cannot be negative, so $169 - y^2 \geq 0$, so $y^2 \leq 169$. Algebraically, we obtain $-13 \leq y \leq 13$, but we discard negative values of y since y represents a physical length. Thus, the domain of $f(y)$ is $\{y : 0 \leq y \leq 13.\}$ In other words, the height of the ladder along the wall cannot exceed the length of the ladder. It is also worth looking at the two extreme values of this domain: $y = 0$ corresponds to the case when the ladder is on the ground; $y = 13$ corresponds to the case when the ladder is totally upright against the wall.

3. Train A travels at 60 miles per hour. Train B travels at 80 miles per hour. The trains leave the same station at the same time. Find the distance between the trains t hours later supposing

(a) train A and train B each travel east.

(b) train A travels east and train B travels west.

(c) train A travels east and train B travels north.

(d) In each case, determine how long until the trains are 50 miles apart.

HINT: You may want to find the distance travelled after 1 hour, then after 2 hours, etc., until you can see the pattern for general time t hours.

Solution: (a) If both trains travel in the same direction, then after one hour, train A has traveled 60 miles but train B traveled 80 miles, so train B is 20 miles ahead. After two hours, the trains have traveled 120 and 160 miles, respectively, and so they are 40 miles apart. After t hours, the trains have traveled $60\,t$ and $80\,t$ miles and so they are $80\,t - 60\,t = 20t$ miles apart. (The moral is that if objects move in the same direction, then the speed at which they seperate is the difference of their individual speeds.)

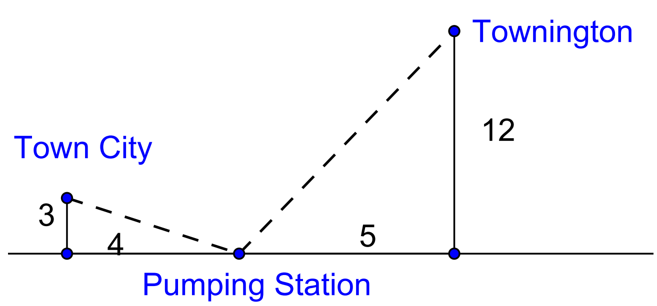
(b) If the trains travel in opposite direction then after one hour, train A has traveled 60 miles but train B traveled 80 miles in the other direction, so the trains are $60 + 80 = 140$ miles apart. After t hours, the trains have traveled $60\,t$ and $80\,t$ miles and so they are $80\,t + 60\,t = 140t$ miles apart. (The moral is that if objects move in the opposite direction, then the speed at which they seperate is the sum of their individual speeds.)

(c) Draw a picture of the locations of the trains. You'll see a right triangle, where the vertices are the two trains and the train station. After t hours, the lengths of the two legs of the right triangle are $60\,t$ and $80\,t$. The distance between the trains is the length of the hypotenuse, which is

$$\sqrt{(60t)^2 + (80t)^2} = \sqrt{3600t^2 + 6400t^2} = \sqrt{10000t^2} = 100t$$

(d) For part (a), solve $20t = 50$ for t to find $t = 50/20 = 2.5$ hours. For part (b), solve $140t = 50$ for t to find $t = 50/140 \approx 0.35$ hours. For part c, solve $100t = 50$ for t to find $t = 50/100 = 0.5$ hours. Be sure to check that these answers make intuitive sense: in part (a) the trains are moving in the same direction so it will take a long time for trains to separate, but in part (b) the trains are moving in opposite directions so the trains separate very quickly.

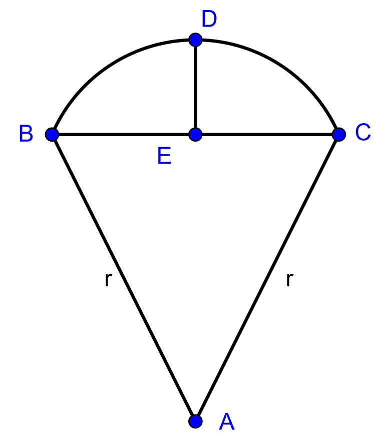
4. The diagram below shows two towns, a river and a pumping station (located on the river). Determine the total length of pipe needed to connect each town to the pumping station.



Solution: Use the Pythagorean Theorem twice to compute the lengths of the two dotted lines. The distance between Town City and the Pumping Station is $\sqrt{3^2 + 4^2} = 5$ miles. The distance between the

Pumping Station and Townington is $\sqrt{5^2 + 12^2} = 13$ miles. Thus, the total length of pipe required is $5 + 13 = 18$ miles.

5. The diagram shows a circular arc, BDC. This arc is part of a circle with center at A and radius r. Determine r, given that length of segment BC is 2 inches and the length of segment DE is 0.05 inches. (The figure is not drawn to scale. Hint: Triangle AEB is a right triangle. How are the lengths of the segments AE and ED related?)



Solution: The key insight is that the distance from A to D is also equal to the radius r. Then, the distance from A to E is $r - DE = r - 0.05$. But the vertices A, B, and E form a right triangle, with legs with lengths $BE = 1$ and $AE = r - 0.05$ and hypotenuse of length $AB = r$, thus

$$r^2 = 1^2 + (r - 0.05)^2 \Longrightarrow r^2 = 1 + r^2 - 0.1r + 0.0025$$

The r^2's cancel, and so $0.1r = 1 + 0.0025$ so $r = 10.025$ inches.

Average and Instantaneous Rates of Change (pages 6-7), Solutions

1. A train travels from city A to city B, pauses, then travels from city B to city C. The train leaves city A at time 10:00 am and arrives at city B at 12:15 pm. The train leaves city B at 2:00 pm and arrives at city C three hours later. The average velocity of the train, while travelling from A to B, was 45 miles per hour. The distance between city B and city C is 240 miles. What is the average velocity of the train from city A to city C (including the stop)?

Solution:
The average velocity is distance traveled divided by time elapsed. Time passes from $10:00$ am to $5:00$ pm, thus 7 hours have elapsed. The distance between B and C is given. The distance between A and B can be found algebraically. The time of travel from A to B is 2.25 hours. During this time, the train travels at 45 miles per hour, so the distance from A to B is $45 \cdot 2.25 = 101.25$ miles. The total distance is therefore $101.25 + 240 = 341.25$ miles and so the average velocity of the whole trip is

$$\frac{341.25}{7} = 48.75 \text{ miles per hour}$$

2. A train leaves city A at 8:00 am and arrives at city B at 10:00 am. The average velocity of the train from A to B was 60 miles per hour. The train leaves city B at 10:00 am and arrives at city C at 1:00 pm. Find the average velocity of the train from city B to C, given that the average velocity from A to C was 50 miles per hour.

Solution:
The time of travel from A to C is 5 hours and so the distance between A and C is $50 \cdot 5 = 250$ miles. The time of travel from A to B is 2 hours and so the distance between A and B is $60 \cdot 2 = 120$ miles. Therefore, the distance between B and C is $250 - 120 = 130$ miles. The time of travel from B to C is 3 hours so the average velocity from B to C is $\frac{130}{3} \approx 43.333$ miles per hour.

3. Let

$$f(x) = \frac{3}{x^2+1}$$

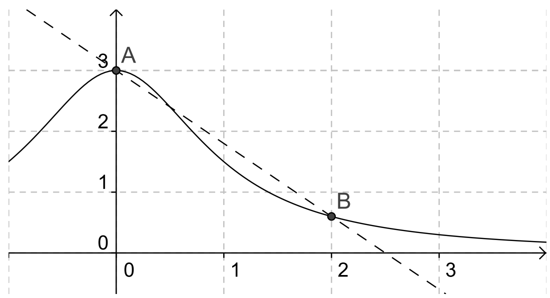
(a) Find the average rate of change of $f(x)$ from $x = 0$ to $x = 2$.

(b) Draw the graph of $y = f(x)$. (A graphing calculator may be useful.) Explain how the rate of change found in part (a) can be represented on this graph.

Solution:
$f(2) = \frac{3}{2^2+1} = \frac{3}{5}$ and $f(0) = \frac{3}{1} = 3$ so the average rate of change is

$$\frac{f(2) - f(0)}{2 - 0} = \frac{(3/5) - 3}{2} = \frac{(-12/5)}{2} = -\frac{6}{5}$$

The average rate of change of $y = f(x)$ from $x = 0$ to $x = 2$ is the slope of the line through the two points, $(0, f(0))$ and $(2, f(2))$. This line is the dotted curve, and $y = f(x)$ is the solid curve.



4. Find a positive number A so that the average rate of change of

$$g(x) = 3x^2 - 1$$

from $x = 2$ to $x = A$ is equal to 33

Solution:
$g(2) = 3 \cdot 2^2 - 1 = 11$ and $g(A) = 3A^2 - 1$ so the average rate of change is

$$\frac{g(A) - g(2)}{A - 2} = \frac{3A^2 - 12}{A - 2} = \frac{3(A^2 - 4)}{A - 2} = \frac{3(A-2)(A+2)}{A - 2} = 3(A + 2)$$

Setting the average rate of change equal to 33,

$$3(A + 2) = 33 \Longrightarrow A + 2 = 11,$$

so $A = 9$.

5. An object is launched up in the air. The height of the object after t seconds is $H(t)$ feet, where $H(t) = -16t^2 + 256t + 64$.

(a) When is the object at its greatest height? (Hint: What must be true about the velocity of the object when it is at the greatest height?)

(b) What is the maximum height of the object?

Solution:
The key to this problem is to realize that at the highest point, the velocity of the ball must be zero. But the velocity is

$$v(t) = H'(t) = 2 \cdot (-16)t + 256 = -32t + 256$$

so we must have

$$-32t + 256 = 0 \Longrightarrow t = 8$$

so the object is at its greatest height after 8 seconds. The maximum height of the object is therefore

$$H(8) = -16 \cdot 8^2 + 256 \cdot 8 + 64 = 1088 \text{ feet}$$

6. Let $g(x) = x^2 - 4x$.

(a) Find the value of x for which the tangent line to $y = g(x)$ has slope equal to 6.

(b) Find the value of $g(x)$ at the point where the tangent line to $y = g(x)$ is parallel to $y = 2x + 5$.

(c) Find a value of x so that the instantaneous rate of change of g at x is equal to the average rate of change of g from $x = -1$ to $x = 3$.

Solution:
The slope of the tangent line to $y = g(x)$ is given by the derivative, $g'(x) = 2x - 4$. For part (a), we need to solve $2x - 4 = 6$ for x, which gives $x = 5$. For part (b), since the tangent line is to be parallel to $y = 2x + 5$, then the slope of these two lines must be equal, so $g'(x) = 2$. Solving $2x - 4 = 2$ for x gives $x = 3$. Thus, the tangent line to $y = g(x)$ is parallel to the line $y = 2x + 5$ when $x = 3$. However, we are asked to find $g(x)$ at this point. Thus, we need to compute $g(3)$, which is

$$g(3) = 3^2 - 4 \cdot 3 = -3$$

For part (c), the required average rate of change of $g(x)$ is

$$\frac{g(3) - g(-1)}{3 - (-1)} = \frac{(3^2 - 4 \cdot 3) - ((-1)^2 - 4 \cdot (-1))}{4} = -2$$

Now we wish to find x so that the instantaneous rate of change $(2x - 4)$ is equal to the average rate of change, (-2), so we need to solve $2x - 4 = -2$ for x, which gives $x = 1$.

7. Suppose $q(x) = 3x^2 - 12x + 8$ and $p(x) = 3x^2 - 12x + 5$.

(a) Find $q'(x)$ and $q'(1)$.

(b) Find the equation of the tangent line to $y = q(x)$ at $x = 1$. Write the equation of the tangent line in slope-intercept form.

(c) Find $p'(x)$ and $p'(1)$.

(d) Find the equation of the tangent line to $y = p(x)$ at $x = 1$. Write the equation of the tangent line in slope-intercept form.

(e) Discussion: What do you notice about your answers? Why is this to be expected, given the graphs of $y = p(x)$ and $y = q(x)$?

Solution:
For parts (a) and (b), $q'(x) = 2 \cdot 3x - 12 = 6x - 12$ and so $q'(1) = -6$. Also, $q(1) = 3 - 12 + 8 = -1$. Thus, the desired tangent line from part (b) is

$$y - (-1) = -6(x - 1) \Longrightarrow y = -6x + 5$$

For parts (c) and (d), $p'(x) = 2 \cdot 3x - 12 = 6x - 12$ and so $p'(1) = -6$. Also, $p(1) = 3 - 12 + 5 = -4$. Thus, the desired tangent line from part (d) is

$$y - (-4) = -6(x - 1) \Longrightarrow y = -6x + 2$$

Now notice the derivatives of $q(x)$ and $p(x)$ are the same. This is to be expected, since $q(x) = p(x)+3$, ie., the graph of $q(x)$ is obtained by shifting the graph of $p(x)$ vertically by 3 units, and therefore the tangent lines to $y = p(x)$ and $y = q(x)$ at any fixed x should be parallel.

Limits (pages 8-9), Solutions

This worksheet focuses on *limits* and the related idea of *continuity*. Many limits can be computed numerically (through a table of values), graphically, and algebraically. When possible, try to compute each of the limits below using all three methods. For a given limit, decide if one method is more efficient than the other methods.

1. Compute each of the limits. Which limits can be computed by evaluating the expression at the limiting value of the independent variable?

$(a)\quad \lim_{t\to 3}(4t+7)$

$(b)\quad \lim_{x\to 2}\dfrac{x^2-5x+6}{x^2-3x+2}$

$(c)\quad \lim_{x\to 1}\dfrac{x^2-5x+6}{x^2-3x+1}$

$(d)\quad \lim_{t\to 0}\left(\dfrac{2}{t}+\dfrac{7t-4}{2t}\right)$

$(e)\quad \lim_{h\to 0}\dfrac{(5+2h)^2-25}{h}$

$(f)\quad \lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

Solution: Each of the above limits can be computed simply by evaluating the expression at the limiting value, unless the denominator tends to zero. Thus, the limits in (a) and (c) can be computed by evaluating the expression, the limits in (b), (d), (e), and (f) will require simplifying the expressions before evaluating.

For (a),

$$\lim_{t\to 3}(4t+7) = 4\cdot 3+7 = 19$$

For (b), factor the numerator and denominators and cancel common terms before evaluating:

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-3x+2} = \lim_{x\to 2}\frac{(x-2)(x-3)}{(x-2)(x-1)} = \lim_{x\to 2}\frac{(x-3)}{(x-1)} = \frac{2-3}{2-1} = -1$$

For (c),

$$\lim_{x\to 1}\frac{x^2-5x+6}{x^2-3x+1} = \frac{1^2-5\cdot 1+6}{1^2-3\cdot 1+1} = \frac{2}{-1} = -2$$

For (d), first find a common denominator so that the two fractions can be combined, then factor and cancel, then evaluate:

$$\lim_{t\to 0}\left(\frac{2}{t}+\frac{7t-4}{2t}\right) = \lim_{t\to 0}\left(\frac{4}{2t}+\frac{7t-4}{2t}\right) = \lim_{t\to 0}\left(\frac{4+7t-4}{2t}\right) = \lim_{t\to 0}\left(\frac{7}{2}\right) = \frac{7}{2}$$

For (e), first expand the square, then collect like terms, factor and cancel, then evaluate:

$$\lim_{h\to 0}\frac{(5+2h)^2-25}{h} = \lim_{h\to 0}\frac{(25+20h+4h^2)-25}{h} = \lim_{h\to 0}\frac{(20+4h)h}{h} = \lim_{h\to 0}20+4h = 20+4\cdot 0 = 20$$

Alternatively, you could have used the difference of squares formula, $A^2-B^2=(A+B)(A-B)$ to simplify the numerator as:

$$(5+2h)^2-25 == (5+2h)^2-5^2 = ((5+2h)+5)\,((5+2h)-5) = (10+2h)(2h)$$

and so

$$\lim_{h\to 0} \frac{(5+2h)^2 - 25}{h} = \lim_{h\to 0} \frac{(10+2h)(2h)}{h} = \lim_{h\to 0} 2(10+2h) = 2(10+2\cdot 0) = 20.$$

For (f), first expand the square, then collect like terms, factor and cancel, then evaluate:

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{(x^2+2xh+h^2) - x^2}{h} = \lim_{h\to 0} \frac{(2x+h)h}{h} = \lim_{h\to 0} 2x+h = 2x+0 = 2x$$

As in part (e), you could have used the difference of squares formula to simplify $(x+h)^2 - x^2$ instead of expanding the square.

2. Each of the limits below is of the form 0/0.

$$(a) \quad \lim_{x\to 2} \frac{x^2-4x+4}{x^2-4} \qquad\qquad (b) \quad \lim_{x\to 2} \frac{x^2-5x+6}{x^2-4x+4}$$

$$(c) \quad \lim_{x\to 2} \frac{x^2-3x+2}{x^2-7x+10} \qquad\qquad (d) \quad \lim_{x\to 2} \frac{x^2-3x+2}{x^2-4}$$

Discussion: In each of the above limits, both the numerator and denominator have $(x-2)$ as a factor, but possibly with different multiplicities. How are the multiplicities related to your final answers?

Solution: The denominators in each of the above limits tends to zero, so it is necessary to factor each of the expressions.

For (a)

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x-2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = \frac{2-2}{2+2} = \frac{0}{4} = 0$$

For (b)

$$\lim_{x\to 2} \frac{x^2-5x+6}{x^2-4x+4} = \lim_{x\to 2} \frac{(x-2)(x-3)}{(x-2)(x-2)} = \lim_{x\to 2} \frac{x-3}{x-2}$$

The denominator still tends to zero as $x \to 2$, and there are no more common factors to cancel with the numerator, so the limit does not exist in this case. (Draw the graph! The vertical asymptote at $x = 2$ confirms that this limit does not exist.)

For (c)

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2-7x+10} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)(x-5)} = \lim_{x\to 2} \frac{x-1}{x-5} = \frac{2-1}{2-5} = -\frac{1}{3}$$

For (d)

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-1}{x+2} = \frac{2-1}{2+5} = \frac{1}{4}$$

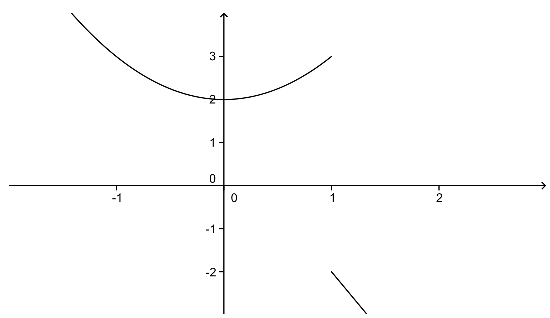
In parts (c) and (d), all of the factors of $x-2$ cancelled. In this case, the limit will be a nonzero number. In part (a), the numerator had more factors of $x-2$ than the denominator, so that the numerator still had a factor of $x-2$ left over even after cancelling. In this case, the limit will be zero. In part (b), the denominator had more factors of $x-2$ than the numerator, so that the denominator still had a factor of $x-2$ left over even after cancelling. In this case, the limit does not exist.

3.

$$f(x) = \begin{cases} x^2+2, & x \le 1 \\ -3x+1, & x > 1 \end{cases}$$

(a)	Sketch the graph of $f(x)$	(b)	Find $f(1)$
(c)	Find $\lim\limits_{x\to 1^-} f(x)$	(d)	Find $\lim\limits_{x\to 1^+} f(x)$
(e)	Find $\lim\limits_{x\to 1} f(x)$	(f)	Is $f(x)$ continuous at $x = 1$?

Solution: We can graph piecewise defined functions by graphing both pieces on the same graph, then erasing the unwanted parts:



For part (b), since the equality holds in the top part of the definition, we use the top function to evaluate $f(1)$:

$$f(1) = 1^2 + 2 = 3$$

For part (c), we use the function in the top part of the definition since $x < 1$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 2 = 1^2 + 2 = 3$$

For part (d), we use the function in the bottom part of the definition since $x > 1$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -3x + 1 = -3 \cdot 1 + 1 = -2$$

For part (e), the limit does not exist since the one sided limits as $x \to 1$ do not agree.

For part (f), $f(x)$ is not continuous at $x = 1$ since the one sided limits as $x \to 1$ do not agree. Alternatively, the jump in the graph at $x = 1$ confirms that $f(x)$ is not continuous at $x = 1$.

4. Use $f(x)$ from the previous problem.

(a)	Sketch the graph of $f(x)$	(b)	Find $f(2)$
(c)	Find $\lim\limits_{x\to 2^-} f(x)$	(d)	Find $\lim\limits_{x\to 2^+} f(x)$
(e)	Find $\lim\limits_{x\to 2} f(x)$	(f)	Is $f(x)$ continuous at $x = 2$?

Solution: We are only concerned with x values near $x = 2$, and so we only use the function in the bottom part of the definition of $f(x)$. Thus,

$$f(2) = -3 \cdot 2 + 1 = -5$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} -3x + 1 = -3 \cdot 2 + 1 = -5$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} -3x + 1 = -3 \cdot 2 + 1 = -5$$

Thus,

$$\lim_{x \to 2} f(x) = -5$$

since the one sided limits agree as $x \to 2$. Finally, $f(x)$ is continuous at $x = 2$ since plugging in $x = 2$ into $f(x)$ gives the same value as the limit of $f(x)$ as $x \to 2$. Alternatively, notice that the graph has no jumps or holes *near* $x = 2$.

5.

$$g(x) = \begin{cases} x - 1, & x < 2; \\ x^2 - A^2, & x \geq 2 \end{cases}$$

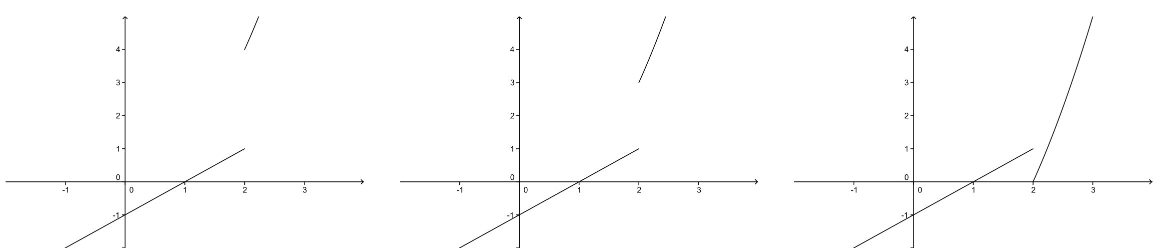
(a) Sketch the graph of $g(x)$ using $A = 0$. Is $g(x)$ continuous?

(b) Sketch the graph of $g(x)$ using $A = 1$. Is $g(x)$ continuous?

(c) Sketch the graph of $g(x)$ using $A = 2$. Is $g(x)$ continuous?

(d) Do you think there is a real value of A which makes $g(x)$ continuous? If so, what is A? If not, why not?

Solution: The graphs for $A = 0$, $A = 1$ and $A = 2$ are shown below.



Each graph has a jump at $x = 2$, so $g(x)$ is not continuous for any of the above values of A. However, for $A = 0$ and $A = 1$, the graph "jumps up" at $x = 2$, whereas for $A = 2$ the graph "jumps down", so we expect $g(x)$ will be continuous for some A between 1 and 2. To guarantee $g(x)$ to be continuous at $x = 2$, we need to require that both pieces in the definition of $g(x)$ agree at $x = 2$. Thus.

$$2 - 1 = 2^2 - A^2$$

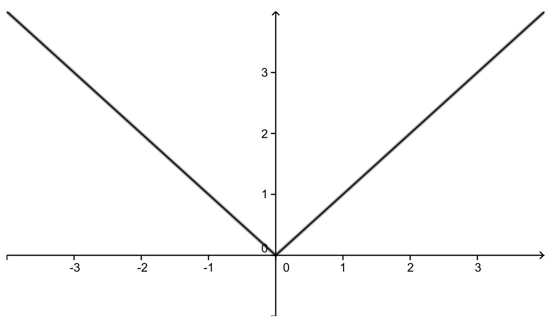
so $g(x)$ will be continuous if $A = \pm\sqrt{3}$.

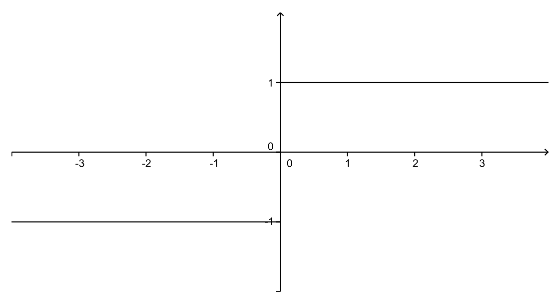
6.

(a) Graph $y = |x|$. Is $y = |x|$ continuous? What is $\lim_{x\to 0} |x|$? If the limit does not exist, then find the one sided limits.

(b) Graph $y = \frac{|x|}{x}$. Is $y = \frac{|x|}{x}$ continuous? What is $\lim_{x\to 0} \frac{|x|}{x}$? If the limit does not exist, then find the one sided limits.

Solution: The function in part (a) is graphed on the left, the function in part (b) is graphed on the right.





For part (a), the graph of $y = |x|$ is a "V" shape, with corner at the origin. As this graph has no jump or hole anywhere, this is a continuous function. Also,

$$\lim_{x\to 0} |x| = 0$$

For part (b), notice that $\frac{|x|}{x}$ is equal to 1 whenever $x > 0$ and $\frac{|x|}{x}$ is equal to -1 whenever $x < 0$, and so the graph consists of two horizontal line segments at heights $y = -1$ and $y = 1$. The graph jumps from $y = -1$ to $y = 1$ at $x = 0$ so $y = \frac{|x|}{x}$ is not continuous. Also

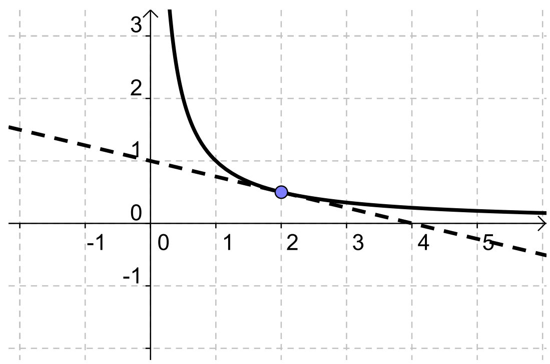
$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$

and

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$

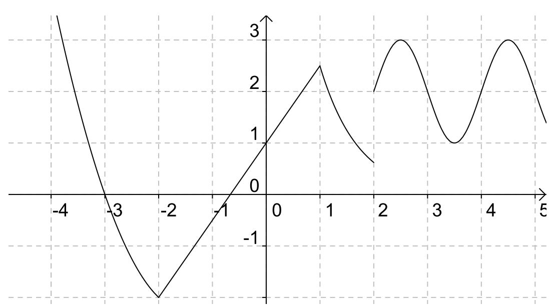
Computing Derivatives With Graphs (page 10), Solutions

1. The graph of the function $y = f(x)$ is shown below, along with the graph of the tangent line to this curve at $x = 2$. Determine $f'(2)$.



Solution: $f'(2)$ is the slope of the tangent line to $y = f(x)$ at $x = 2$, but this line is the dotted line in the picture. This line goes through the points $(0, 1)$ and $(4, 0)$ and so the slope is $-1/4$. Thus, $f'(2) = -1/4$.

2. Determine the x coordinates of all points of nondifferentiability for the function graphed below.



Solution: Graphically, points of non-differentiability arise whenever (1) the graph has a jump or a hole, (2) the graph has a sharp corner, OR (3) the graph has a vertical tangent line. The above graph has corners at $x = -2$ and $x = 1$. The graph has a jump at $x = 2$. Thus, the graph has 3 points of nondifferentiability.

3. Let $g(x) = |x^2 + 2x - 15|$. Determine all points where $g(x)$ is not differentiable.
Solution: Notice that $g(x)$ factors as

$$g(x) = |(x-3)(x+5)|.$$

Since $x^2 + 2x - 15$ is differentiable everywhere and the absolute value function is differentiable everywhere except at the origin, it follows that the only potential points of nondifferentiability of $g(x)$ are at the zeros of $x^2 + 2x - 15$. But

$$x^2 + 2x - 15 = (x-3)(x+5)$$

so the zeros of $x^2 + 2x - 15$ are $x = 3$ and $x = -5$.

A quick look at the graph of $y = g(x)$ verifies that $g(x)$ is not differentiable at $x = -5$ and $x = 3$.